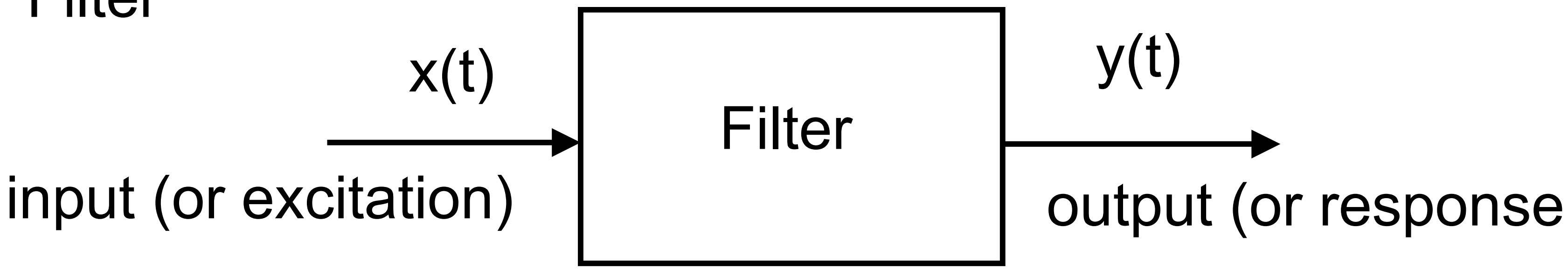
Fundamental Concepts

- Filter



- Fourier series expansion:
  – periodic signal x(t) of period $\frac{2\pi}{\omega_0}$

$$x(t) = a_0 + \sum_{k=1}^{\infty}(a_k \cos k\omega_0 t + b_k \sin k\omega_0 t)$$

$$= A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \phi_k)$$

$$= \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t}$$

where X_k represents the discrete spectrum of x(t)

Fundamental Concepts(Cont.)

– x(t) is not periodic

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{jwt}d\omega$$

where $X(j\omega)$ represents the continuous spectrum of x(t)

- Physical meaning

1. Spectrum-shaping where number X_k or function X(jw) are altered in certain way in order to produce desired form of output signal y(t).

2. If the filter is linear, the harmonic content can not be richer than that of the input signal

3. Desired filter operation can be performed by the appropriate interconnection of elements with chosen values.

Historical Review

Example : Bell system progress in filter technology of voice-frequency (f < 4kHz) application over a nearly 60-years span(1920~1980)

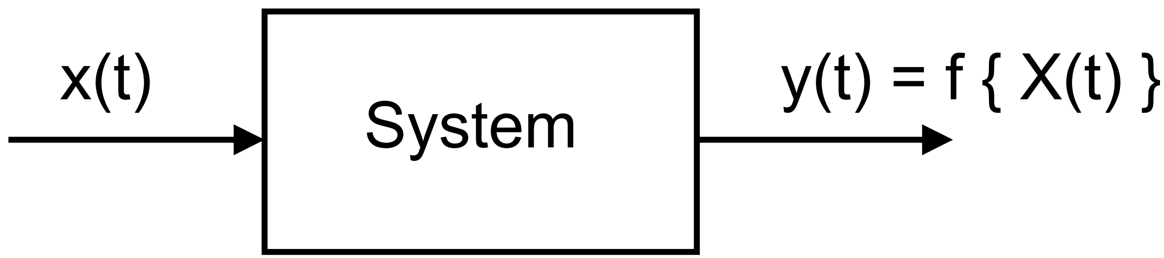
1920 - Passive LC (1)
1969 - Discrete active RC (1)
1973 - Thin film active RC (1)
1975 - Active RC DIP (1)
1980 - Switched-capacitor building block (11)
Digital signal processor (37)

(N) No. of biquardratic section

System classification

- Black box representation:
  – single-input, single-output system is a special case. For a filter, its enough.

input (or excitation) x(t) ; output (or response) y(t)



Black box

  – For a filter, x(t) and y(t) are electrical signals, e.g. voltage, current, or charge.
  – Filter is composed of lumped active and passive elements.
  – A lumped element is defined as one having physical dimensions small compared to the wavelength of the applied signals.

System classification(Cont.)

- System classification

  1. linear and nonlinear systems.
  2. continuous-time and discrete-time (or sampled-data) systems.
  3. time-invariant and time-varying systems.
     (Linear time-invariant is abbreviated as LTI)

- A system is linear if superposition principle is satisfied.
  – superposition

$$y_1 = f(x_1) \quad ; \quad y_2 = f(x_2) \quad ; \quad x = \alpha x_1 + \beta x_2$$

$$y = f(x) = f(\alpha x_1 + \beta x_2) = \alpha f(x_1) + \beta f(x_2) = \alpha y_1 + \beta y_2$$

  – A linear system can be described by a linear differential or difference equation .

  – For a filter , nonlinearity must be eliminated or minimized.
    e.g. overdrive an amplifier => nonlinearity occurs

Continuous-Time and Discrete-Time

- Continuous-time

  Input and output are continuous functions of the continuous variable time.

  $x = x(t)$ & $y = y(t)$; t is time

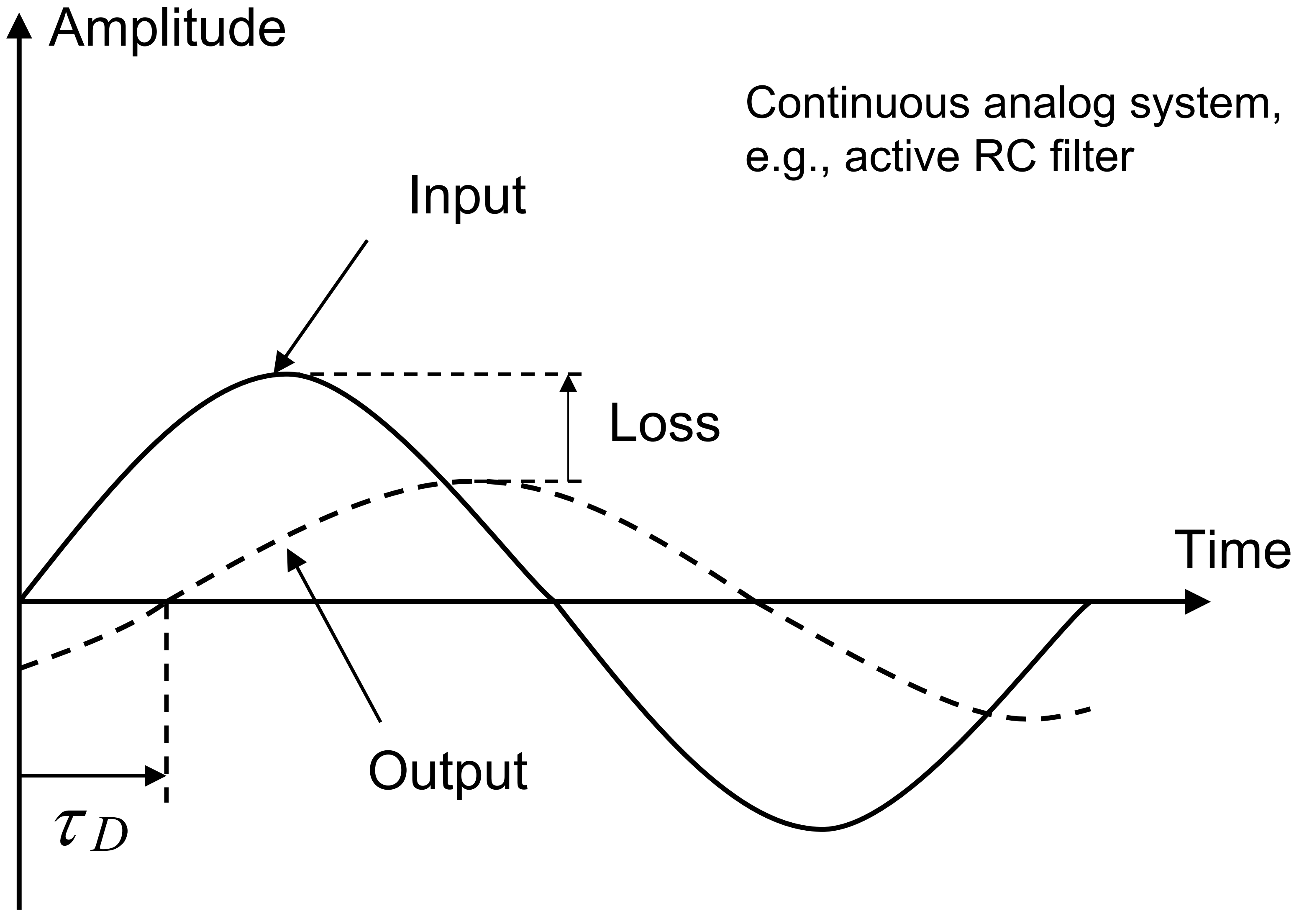
- Discrete-time

  Input and output change at only discrete instants of time. (e.g. sampling instants)

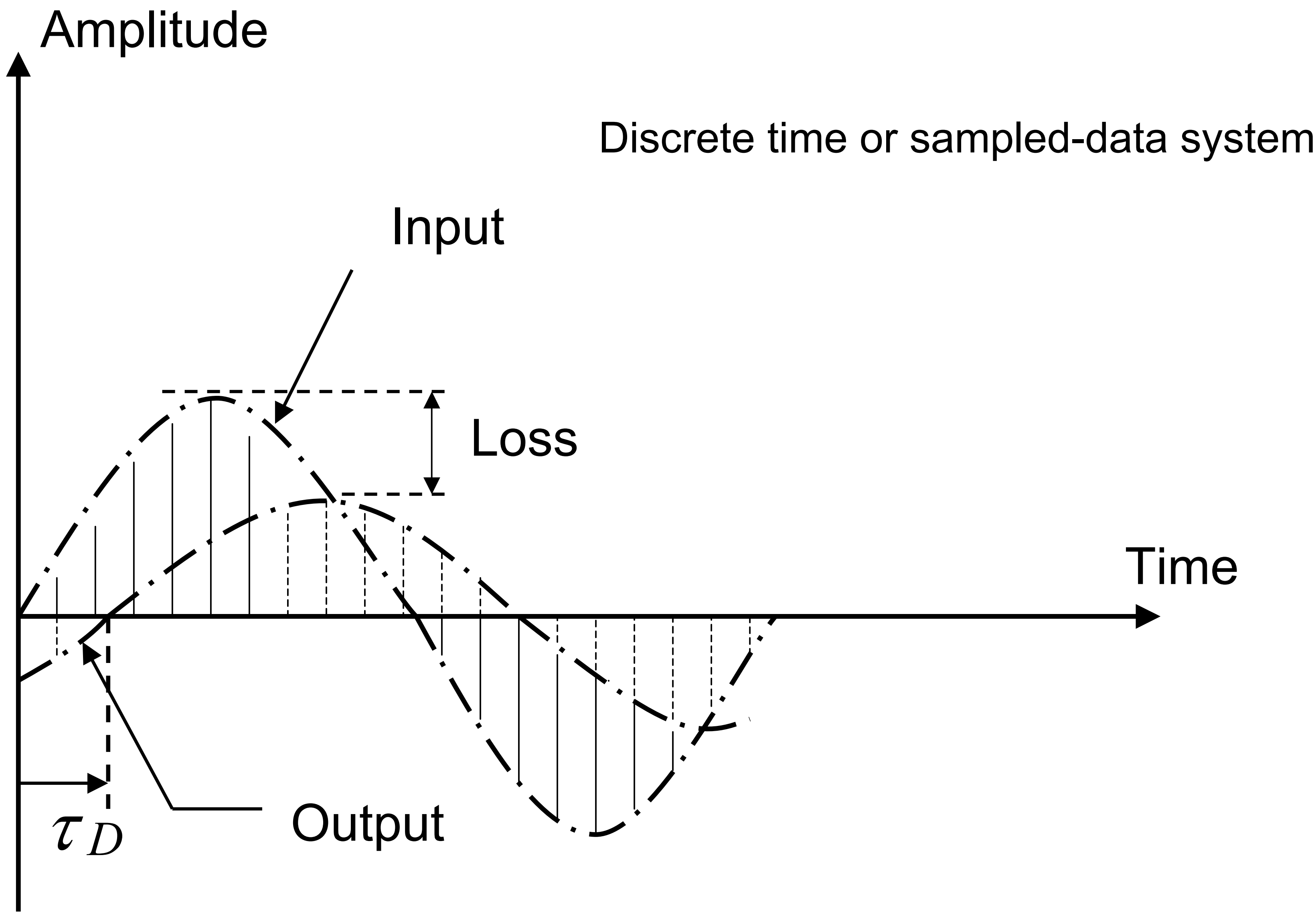
  $x = x(kT)$ & $y = y(kT)$; where k is an integer and T is the time interval between samples

- Mathematical distinction
  - Continuous-time systems are characterized by differential equations.
  - Discrete-time systems are characterized by difference equations.

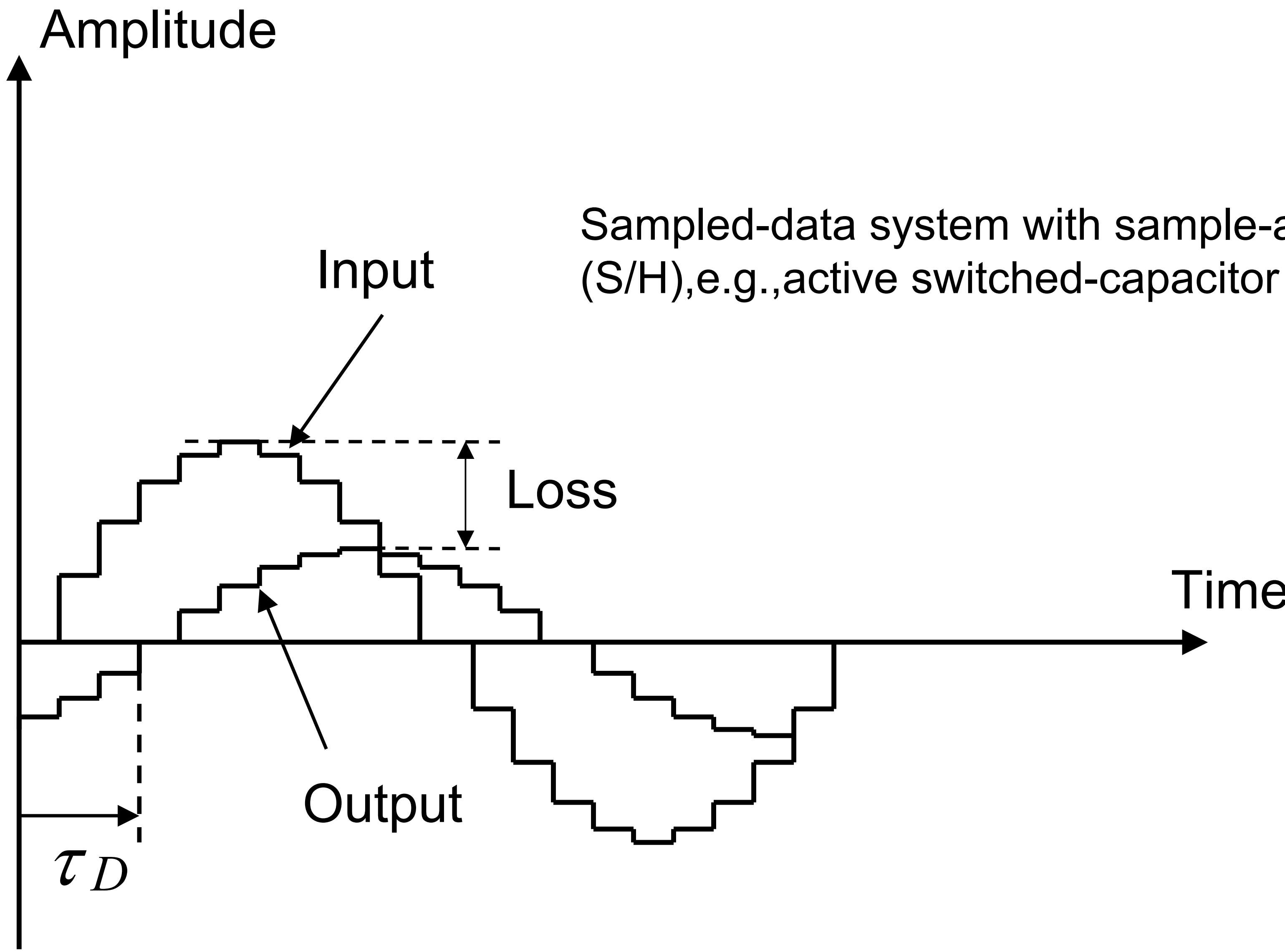
Continuous-Time and Discrete-Time (cont.)



Continuous-Time and Discrete-Time (cont.)



Continuous-Time and Discrete-Time(cont.)



Sampled-data system with sample-and-hold (S/H),e.g.,active switched-capacitor filter

Time-Invariant and Time-Varying Systems

- Time-invariant
  - Mathematical characteristic

A. Continuous-time systems

$x(t) \Rightarrow y(t)$

$x(t - \tau) \Rightarrow y(t - \tau)$

for all $x(t)$ and all τ

B. Discrete-time systems

$x(kT) \Rightarrow y(kT)$

$x[(k-n)T] \Rightarrow y[(k-n)T]$

for any $x(kT)$ and n

  - Physical meaning

System response depends only on the shape of input and not on the time of application .

Causal System

- Response can't precede the excitation

$$\left.\begin{array}{l} \text{if } x(t)=0 \text{ for } t < t_0 \text{ or } mT \\ \text{then } y(t)=0 \text{ for } t < t_0 \text{ or } mT \end{array}\right\} \text{ for all } t_0 \text{ or } mT$$

Representations of Continuous-LTI Systems

- Single-input, single-output, continuous LTI systems
  - Input/output relationship (linear differential equation)

$$b_n \frac{d^n y(t)}{dt^n} + b_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + ... = a_m \frac{d^m x(t)}{dt^m} + a_{m-1} \frac{d^m x(t)}{dt^{m-1}} + ...$$

  where (i) y(t) is output

  (ii) a_i and b_j are real and depend on the network elements

  (a) LTI: a_i and b_j are constant.

  (b) nonlinear: a_i and b_j are functions of x and/or y.

  (c) time-dependent: a_i and b_j are functions of time.

  - If x(t) and initial conditions $y(0), \frac{dy(0)}{dt}, ..., \frac{dy^{n-1}(0)}{dt^{n-1}}$ are known, then y(t) is completely determined.

Representations of Continuous-LTI Systems(Cont.)

- Zero-input response

  Response obtained when the input is zero.
  (Response is not necessarily zero because initial conditions may not be zeros)

- Zero-state response

  Response obtained for any arbitrary input when all initial conditions are zero .

- For a linear system, the complete response is equal to the sum or superposition of the zero-input and zero-state responses.

Frequency-Domain Concepts

- Laplace transform techniques can be used to transfer time-domain differential equations into frequency domain equations, e.g.

$$L\left\{\frac{d^n y(t)}{dt^n}\right\} = S^n Y(s) - S^{n-1} y(0) - S^{n-2}\frac{dy(0)}{dt} .... - \frac{d^{n-1} y(0)}{dt^{n-1}}$$

Hence,
$$b_n \frac{d^n y(t)}{dt^n} + b_{n-1}\frac{d^{n-1} y(t)}{dt^{n-1}} + ... + b_0 y_0(t)$$

$$= a_m \frac{d^n x(t)}{dt^n} + ... + a_0 x_0(t)$$ (for a LTI system)

can be transformed into

$$(b_n S^n + b_{n-1} S^{n-1} + b_{n-2} S^{n-2} ...... + b_0) Y(s) + IC_y(s)$$

$$= (a_m S^m + a_{m-1} S^{m-1} + a_{m-2} S^{m-2} ...... + b_0) X(s) + IC_x(s)$$

where $IC_y(s)$ and $IC_x(s)$ are from initial conditions of y and x. X(s) and Y(s) are excitation and zero-state response.

Frequency-Domain Concepts(Cont.)

- Transfer function H(s) of network

$$H(S)=\frac{L(zero-state\ \ response\ \ \ y(t))}{L(excitation\ \ \ \ x(t))}=\frac{Y(S)}{X(S)}=\frac{a_m S^m+....}{b_n S^n+....}=\frac{N(S)}{D(S)}$$

where $\mathrm{m}\le\mathrm{n}$ for any realizable practical network.

  – Transfer function

$$\frac{V_{out}(s)}{V_{in}(s)}$$; voltage transfer function

$$\frac{V_{out}(s)}{I_{in}(s)}$$; impedance transfer function

$$\frac{I_{out}(s)}{V_{in}(s)}$$; admittance transfer function

  – Driving-point impedance and admittance functions

$$Z_{in}(s)=\frac{\mathrm{V_{in}}(S)}{I_{in}(S)}\quad \&\quad \mathrm{Y_{in}}(s)=\frac{I_{in}(S)}{V_{in}(S)};\ \ where\quad Y_{in}(s)=\frac{1}{\mathrm{Z_{in}}(s)}$$

Frequency-Domain Concepts(Cont.)

- System analysis

  1. Time-domain differential equation.
  2. Frequency-domain equation(i.e. S-domain equation).

     (2 is proved to be a more convenient method from experience).

- Transfer function of continuous LTI
  - A ratio of two polynomials in S with real coefficients.
  - Can be factored as

$$H(S) = \frac{N(S)}{D(S)} = \frac{a_m (S - Z_1)(S - Z_2)...(S - Z_n)}{b_n (S - P_1)(S - P_2)...(S - P_n)}$$

where Z_i are zeros (H(S)=0 when S = Z_i)

P_i are poles (D(S)=0 when S = P_i)

  - Z_i , P_i = σ +jw on complex S-plane.

Frequency-Domain Concepts(Cont.)

– Transfer function with real coefficient
  ⇨ Poles and zeros are real or conjugate pairs (complex or imaginary).
– For stability, all poles must lie in the left plane.
  (i.e. D(s) is a Hurwitz polynomial)
– Poles lie in the left plane
  ⇨ zero input response decays with time
– Poles lie on the jw-axis
  ⇨ the network oscillates
– Poles lie in the right plane
  ⇨ responses grow exponentially with time
– when zero of N(S) lie on or to the left of the jw-axis. (i.e. there are no right-plane zeros), H(S) is referred to as a minimum-phase function.

Time-Domain Concepts

- Continuous LTI systems (system representation)

A. differential equation

$$b_n \frac{d^n y(t)}{dt^n} + b_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + ... = a_m \frac{d^m x(t)}{dt^m} + a_{m-1} \frac{d^{m-1} x(t)}{dt^{m-1}} + ...$$

B. convolution or superposition integral

– convolute excitation x(t) with impulse response h(t) to obtain y(t)

$$y(t) = \int_0^t h(\lambda) x(t - \lambda) d\lambda$$

where λ is a dummy integral variable and we assume that the system is causal and x(t)=0 for t<0.

Time-Domain Concepts(Cont.)

– h(t) is impulse response

Assume $x(t) = \delta(t)$, then

$$y(t) = \int_0^t h(\lambda)\delta(t-\lambda)d\lambda = h(t)\int_0^t \delta(t-\lambda)d\lambda = h(t)$$

(definition : $\int_0^t \delta(t-\lambda)d\lambda = 1$ with $\delta(t-\lambda) = 0$ for $\lambda \neq t$)

– convolution v.s. frequency-domain representation

$$Y(S) = L\{y(t)\} = \int_0^t e^{-st}[\int_0^t x(t-\lambda)h(\lambda)d\lambda]dt = H(S)X(S)$$

where transfer function $H(S) = L\{y(t)\} = \int_0^\infty h(t)e^{-st}dt$

Ideal Distortionless Transmission

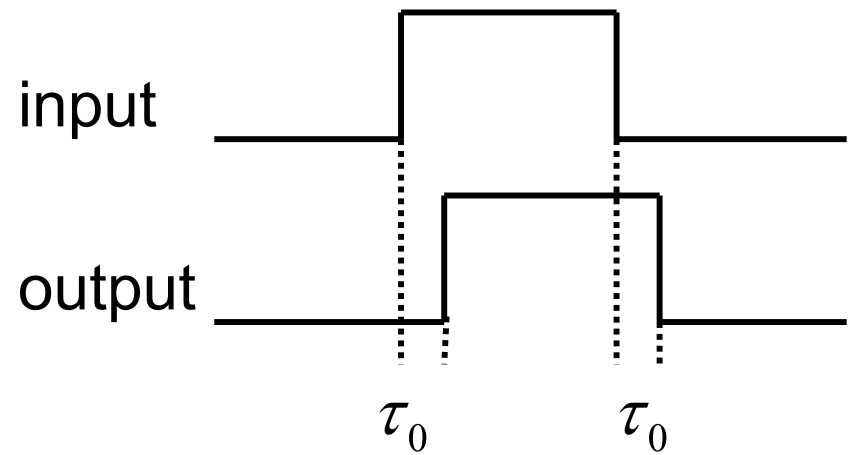
- y(t) is perfect replica of x(t); may be with amplification k and delay τ_0 .

  $y(t) = kx(t - \tau_0)$

  ⇨ frequency domain

  $Y(S) = L\{y(t)\} = ke^{-S\tau_0} X(S)$

  ⇨ $H(S) = Ke^{-S\tau_0}$



  – Where 1. H(S) has { constant magnitude K; linear phase $\phi = -\omega\tau_0$ }

  2. H(S) is not a real rational function

  ⇨ not realizable as a lumped network with a finite number of elements.

  3. group delay $\tau(\omega) = \tau_0 = constant$

Ideal Distortionless Transmission(Cont.)

– From 2, method for approximating the above $H(S) = Ke^{-S\tau_0}$ by a rational function must be developed such that it becomes realizable physically.

– Approximation method results in transmission errors since any physical network has in practice frequency-dependent magnitude and delay .

– Two method to define deviations from an ideal transmission:

1. step response.
2. impulse response.

Step Response

- Ideal transmission

  $$H(S) = Ke^{-S\tau_0}$$

  where H(S) is a physically unrealizable transfer function

  step input : $x(t) = u(t) \Rightarrow X(S) = \frac{1}{S}$

  output step function(step responses): a(t)

  $$\Rightarrow y(t) = a(t) = L^{-1}\{H(S)X(S)\} = ku(t - \tau_0)$$

- Delay time τ_d & rise time τ_r :

  τ_d : time required for step response to reach 50% of its final value

  τ_r : time required for step response to rise for 10% to 90%

  γ : overshoot

Step Response(Cont.)

- Example: one-pole case

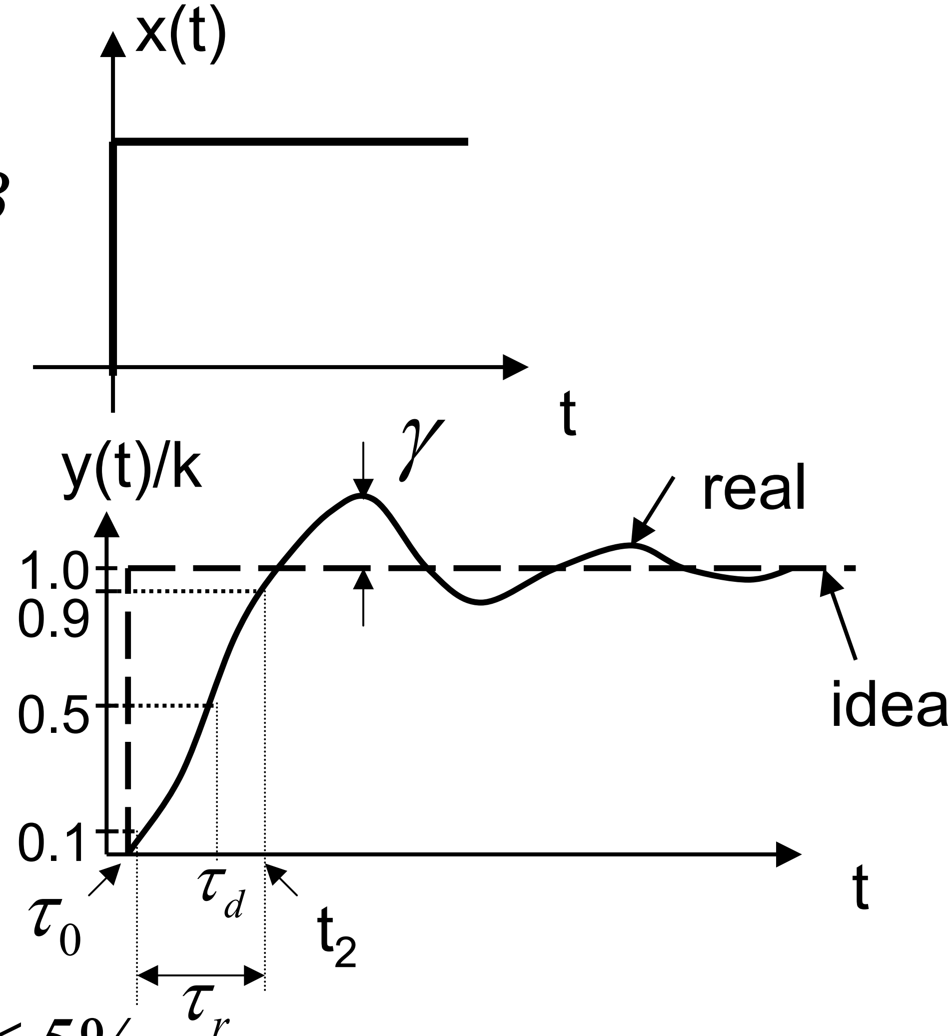
$$H(S)=\frac{1}{S+a_1}\ ;\ \ a_1=\omega_{-3dB}$$

IF x(t)=u(t)

$$\Rightarrow\ y(t)=(1-e^{-a_1 t})u(t)$$

– no over shoot $\gamma = 0$

$$\tau_d \approx \frac{0.69}{a_1}\ \ \&\ \ \tau_r \approx \frac{2.2}{a_1}$$



- In general for overshoot $\gamma \le 5\%$

$$\tau_r\omega_{-3dB} \approx 2.2\ \ \text{(i.e. } \tau_r f_{-3dB} \approx 0.35\ \text{)}$$

Impulse Response

- Ideal transmission

$$X(t) = \delta(t)$$

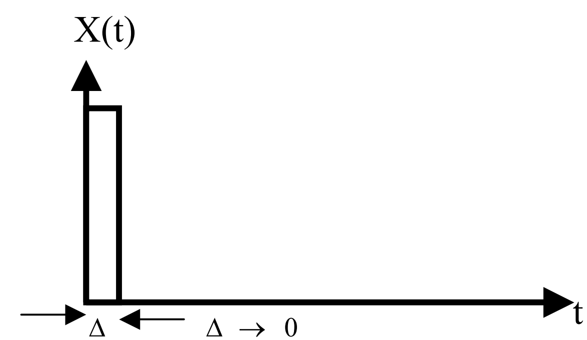
$$H(s) = ke^{-S\tau_o} \quad \text{; unrealizable}$$

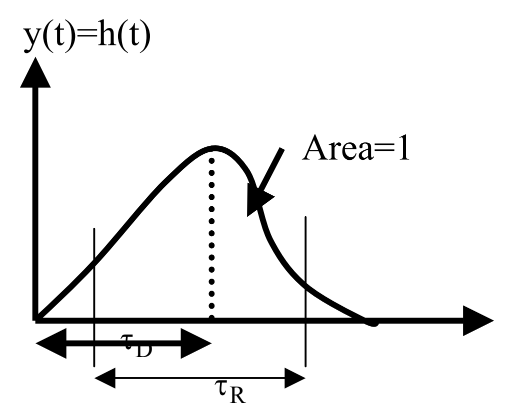
$$y(t) = h(t) = k\delta(t - \tau_0)$$

  - Impulse response $h(t) = k\delta(t - \tau_0)$
  - step response $a(t) = ku(t - \tau_0)$

$$=> h(t) = \frac{da(t)}{dt}$$

- One can obtain impulse response from step response, and vice versa .





Calculation of τ_D and τ_R

- Precise calculation of τ_D and τ_R are usually time-consuming
- Convenient method resulting in considerable simplification is proposed by Elmore(Assuming negligible overshoot or none)

  - Elmore's definition

$$\tau_D = \int_0^\infty t \; h(t)dt$$

$$\tau_R = [2\pi\int_0^\infty (t-\tau_D)^2 h(t)dt]^{1/2} = \sqrt{2\pi}[\int_0^\infty t^2 h(t)dt - \tau_D^2]^{1/2}$$

- Consider the normalized transfer function (H(0)=1)

  - H(s)= $\dfrac{1 + a_1 s + a_2 s^2 + ... + a_m s^m}{1 + b_1 s + b_2 s^2 + ... + b_n s^n} \quad n \geq m$
  - by direct division $H(S) = 1 - (b_1 - a_1)s + (b_1^2 - a_1 b_1 + a_2 - b_2)s^2 + ...$ (1)
  - from impulse response

$$H(s) = \int_0^\infty h(t)e^{-st}dt = \int_0^\infty h(t)(1 - st + \frac{s^2 t^2}{2!} - ....)dt$$

(by incorporating Elmore's definitions)

$$= 1 - s\tau_D + \frac{s^2}{2!}(\frac{\tau_R^2}{2\pi} + \tau_D^2) - .. \qquad (2)$$

Calculation of τ_D and τ_R (Cont.)

- Equating (1) and (2) yields

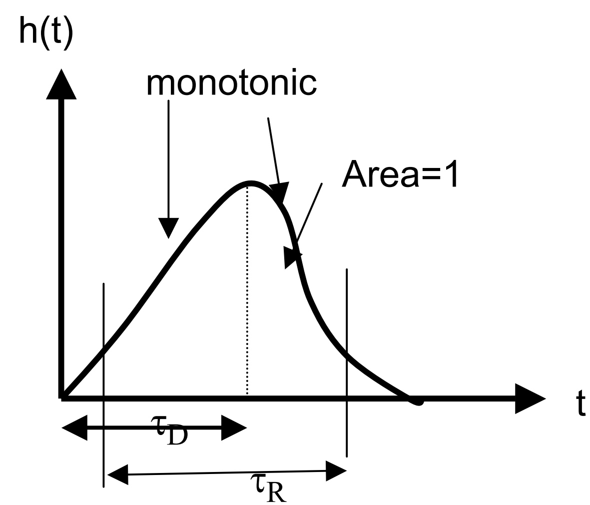
$$\tau_D = b_1 - a_1 \quad \text{---} \quad (3)$$

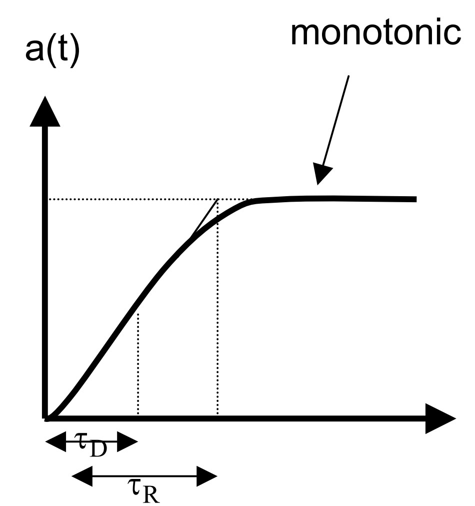
$$\tau_R = \{2\pi[b_1^2 - a_1^2 + 2(a_2 - b_2)]\}^{1/2} \quad \text{------------------------------------} \quad (4)$$

- Ease of computation using Elmore's definition
  - For higher-order systems with no overshoot (i.e.monotonic) can be decomposed to K monotonic cascaded stages.

$$(\tau_D)_{total} = \sum_{i=1}^{k} (\tau_D)_i$$

$$(\tau_R)_{total} = [\sum_{i=1}^{k} (\tau_R)_i^2]^{1/2}$$





- Example 1-2

Normalization

- Advantages

  1. avoid the tedium of having to manipulate large power of 10
  2. minimize the effect of roundoff errors.

- Normalization

  1. Frequency normalization

     -Frequency scale changed by dividing the frequency variable by a conveniently chosen normalization frequency Ω_0

     Normalization equation $S_n = \frac{S}{\Omega_0}$

  2. Impedance normalization

     -by dividing all impedances in the circuit by a normalization resistance R_0. $R_n = \frac{R}{R_o}$, $C_n = C_o R_o$, $L_n = \frac{L}{R_o}$

     -Normalization equation

Normalization(Cont.)

$$R_n = \frac{R}{R_0}\ ,\ S_n L_n = \frac{SL}{R_0} = \frac{(S/\Omega_0)\Omega_0 L}{R_0}\ ,\ \frac{1}{S_n C_n} = \frac{1}{SCR_0} = \frac{1}{(S/\Omega_0)\Omega_0 C R_0}$$

$$\Rightarrow R_n = \frac{R}{R_0}\ ,\ L_n = L\frac{\Omega_0}{R_0}\ ,\ C_n = C\Omega_0 R_0$$

- The actual unnormalized physical parameters R, L, and C are obtained by inverting the normalization equations.
- Comments(practical concerns):

  1. normalization is to remove dimensions

     (S_n, R_n, L_n and C_n are dimensionless)

     =>Easy remember.

  2. Dimensionless network

=>designer can choose convenient and practical element values.

- Example 1-3

Type of Filters

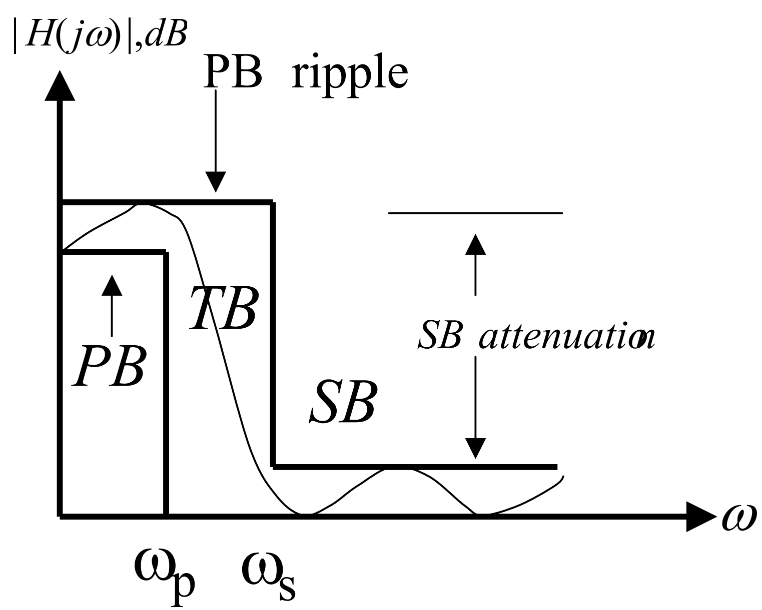
- Five major types
  1. Lowpass filter
  2. Highpass filter
  3. Bandpass filter
  4. Bandreject filter

  } Magnitude spec. are of primary concern

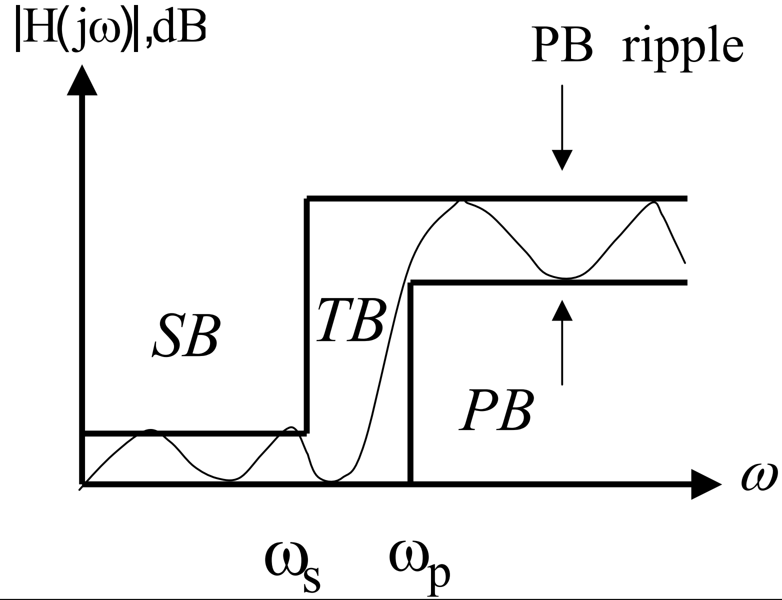
  5. Allpass filter--phase or delay specs are of primary concern.

- Filter magnitude specifications

  -Lowpass filter

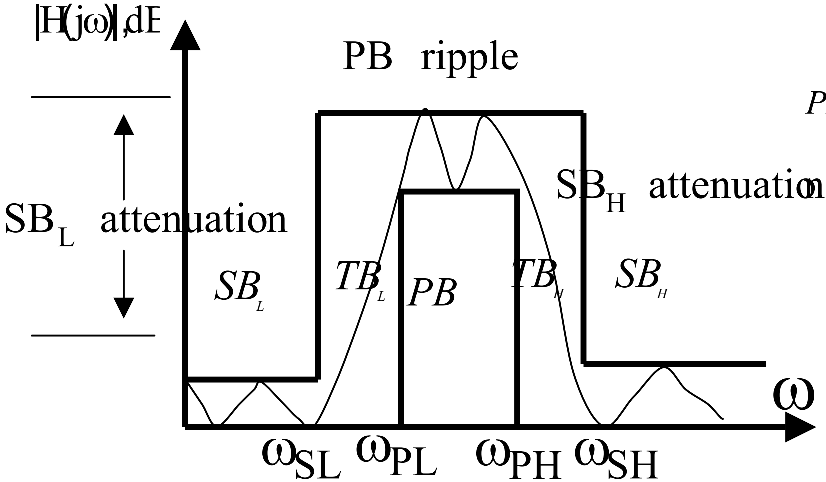


  -Highpass filter

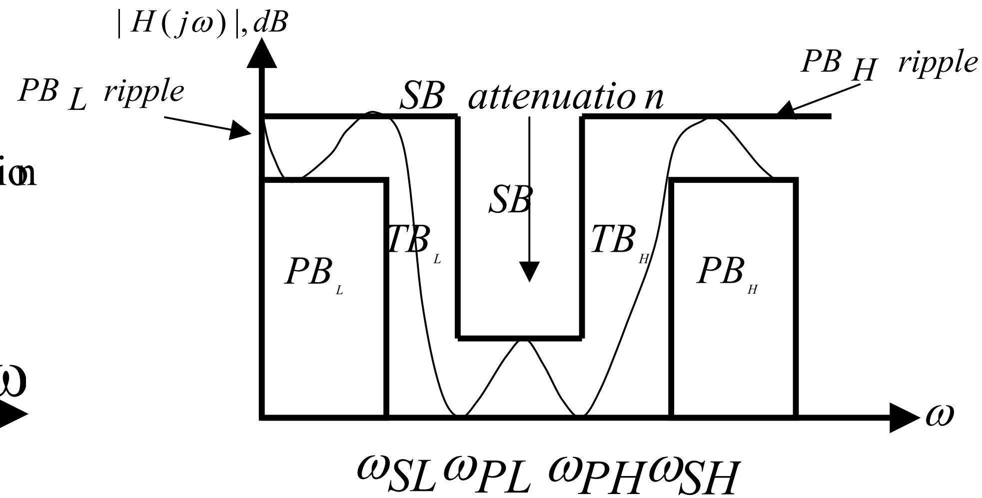


Type of Filters(Cont.)

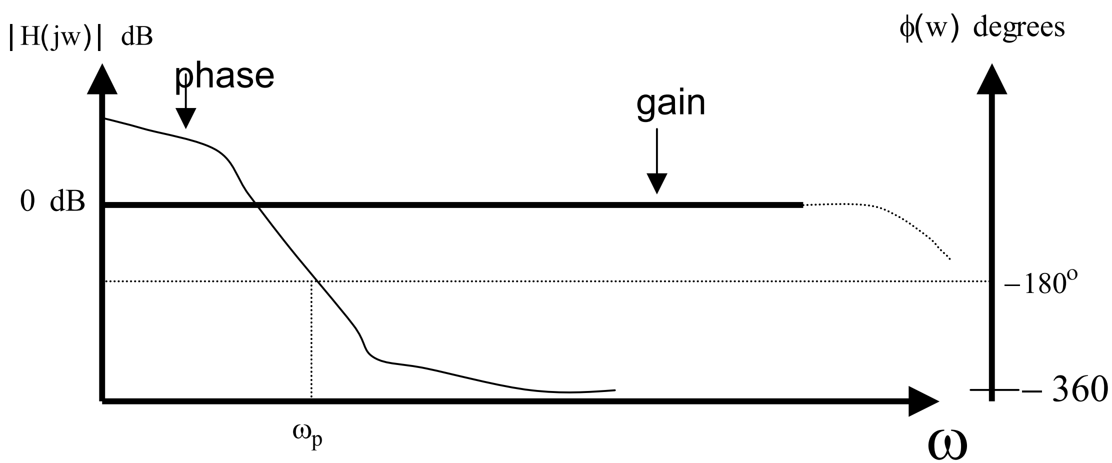
-Bandpass filter



-Bandreject filter



-All pass filter



Filter Phase or Delay Specs.

- Frequency dependent delay

 - Usually not important for voice or audio.(Human ear is very insensitive to phase change with frequency.)

 - Can cause intolerable distortion in video or digital transmission
 =>Nonminimum phase function may be needed
 - Minimum phase function：with only left half-phase zeros

- Examples :
 1. Realization of nonminimum phase function

 (i) Let $H_N(s) = H_M(s)H_{AP}(s)$
 where $|H_M| = |H_N|$

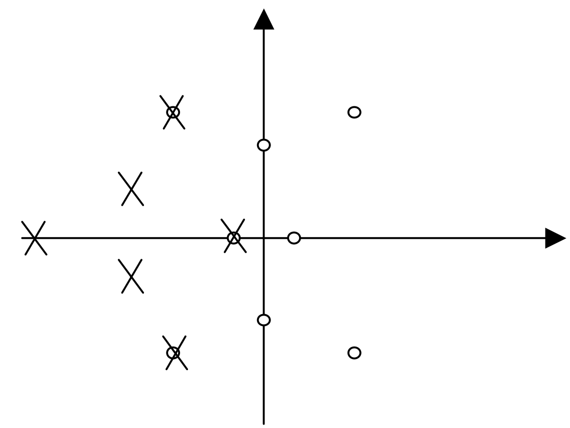
 N: nonminimum phase

 M: minimum phase

 AP: all-pass

Filter Phase or Delay Specs.(Cont.)

(ii) Augment pole-zero



(iii) $H_{AP} = \frac{N_{AP}(s)}{D_{AP}(s)}$ is a allpass function

where N_{AP} is formed by all right-plane zeros

D_{AP} is formed by all left-plane poles which are mirror images of the right-plane zeros

Hence , $N_{AP} = \pm D_{AP}(-s)$

$$H_{AP} = \pm \frac{D_{AP}(-s)}{D_{AP}(s)}$$

$$\text{phase } \phi_{AP} = -2\tan^{-1}\frac{D_I(w)}{D_R(w)}$$

where $D_I(w) = \text{Im}[D_{AP}(jw)]$

$$D_R(w) = \text{Re}[D_{Ap}(jw)]$$

$$\text{Delay } \tau(w) = -\frac{d\phi_{AP}(w)}{dw}$$

Filter Phase or Delay Specs.(Cont.)

-Using $H_{AP}(s)$, a desirable delay function without any effect on the magnitude can be achieved.

-Example

$$H_T(s) = H(s)H_{AP}(s)$$

where $|H_T(s)| = |H(s)|$

$$\phi_T(w) = \phi(w) + \phi_{AP}(w)$$

$$\tau_T(w) = \tau(w) + \tau_{AP}(w)$$

- The cascaded allpass can, of course, only increase the phase and delay of H(s); this is normaly no problem, because for distortionless transmission, only the linearity of ϕ_T i.e., the constancy of τ_T , in the frequency range of interest is important, not its actual size.

- Example 1-4

Filter Phase or Delay Specs.(Cont.)

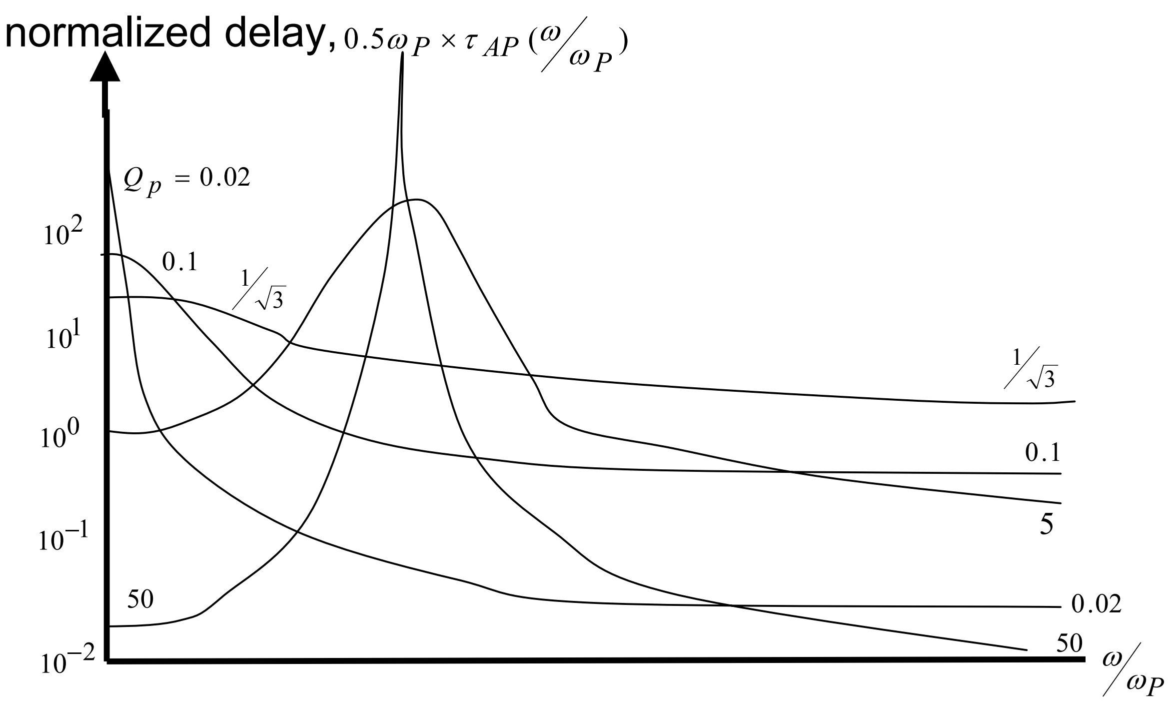
2. Phase or delay equalization

(i) make total delay as flat as possible in the frequency range of interest.

(ii) obtain prescribed delay

-Precision design requires computer aids.

-Uncritical design of low order($\frac{\Delta\tau}{\tau} \approx 10 \sim 20\%$) can be performed manually with the aid of the curves below.



Second Order Filters

- 2nd-order transfer function

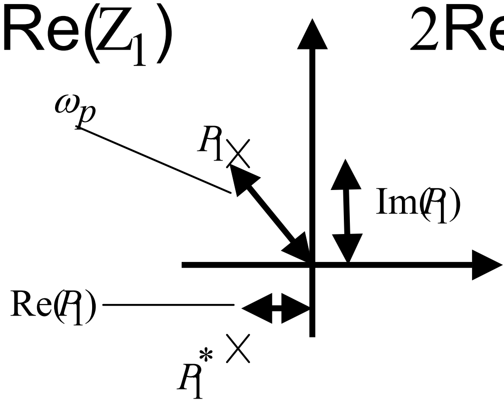
$$H(s)=\frac{a_2S^2+a_1S+a_0}{S^2+b_1S+b_0}=\frac{a_2(S+Z_1)(S+Z_2)}{(S+P_1)(S+P_2)}$$

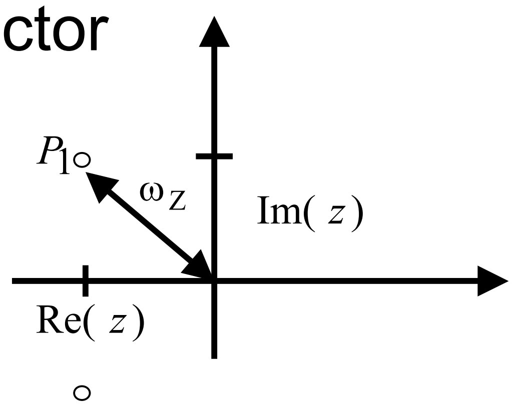
; where P&Z are real or complex pairs

$$=K\frac{S^2+[2Re(Z_1)]S+Re(Z_1)^2+Im(Z_1)^2}{S^2+[2Re(P_1)]S+Re(P_1)^2+Im(P_1)^2}=K\frac{S^2+(\frac{\omega_z}{Q_z})S+\omega_z^2}{S^2+(\frac{\omega_p}{Q_p})S+\omega_p^2}$$

where $Q_p=\frac{\omega_p}{2Re(P_1)}=\frac{\sqrt{[Re(P_1)]^2+[Im(P_1)]^2}}{2Re(P_1)}$ is the pole quality factor

$Q_z=\frac{\omega_z}{2Re(Z_1)}=\frac{\sqrt{[Re(Z_1)]^2+[Im(Z_1)]^2}}{2Re(Z_1)}$ is the zero quality factor





Second Order Filters(Cont.)

$$H_{LP}(s) = K\frac{\omega_p^2}{S^2 + (\frac{\omega_p}{Q_p})S + \omega_p^2}; \quad H_{HP}(s) = K\frac{S^2}{S^2 + (\frac{\omega_p}{Q_p})S + \omega_p^2}$$

$$H_{BP}(s) = K\frac{(\frac{\omega_p}{Q_p})S}{S^2 + (\frac{\omega_p}{Q_p})S + \omega_p^2}; \quad H_{BR}(s) = K\frac{S^2 + \omega_z^2}{S^2 + (\frac{\omega_p}{Q_p})S + \omega_p^2}$$

$$H_{AP}(s) = K\frac{S^2 - (\frac{\omega_p}{Q_p})S + \omega_p^2}{S^2 + (\frac{\omega_p}{Q_p})S + \omega_p^2} = K\frac{S_n^2 - (\frac{S_n}{Q_p}) + 1}{S_n^2 + (\frac{S_n}{Q_p}) + 1}$$

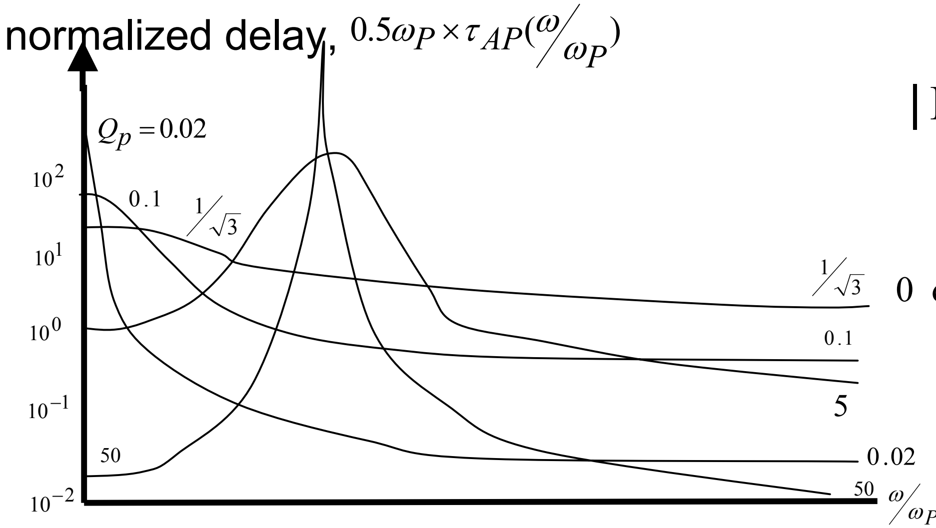
where $S_n = \frac{S}{\omega_p}$ & $\omega_n = \frac{\omega}{\omega_p}$; n means normalized

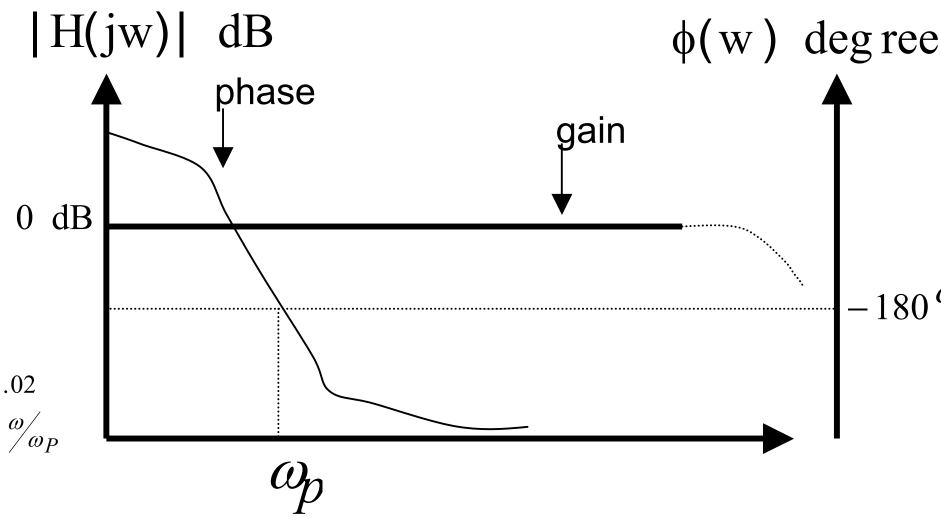
Second Order Filters(Cont.)

– Normalized phase $\phi_{AP}(\omega_n) = -2\tan^{-1}\left(\frac{\omega_n / Q_p}{1-\omega_n^2}\right)$

– normalized delay

$$\tau_{n,AP}(\omega_n) = \omega_p \tau_{AP}(\omega_n) = \frac{2}{Q_p}\frac{1+\omega_n^2}{(1-\omega_n^2)^2 + (\omega_n / Q_p)^2}$$





Approximation Methods

- Distortionless filter is not realizable as discussed before.
  (i.e. Ideal transfer characteristic are not realizable)
- Practical realization
  1. H(s) must be a real rational function such that it can be realized by lumped circuits as discussed before.
  2. H(s) is only a approximation of ideal characteristics on both magnitude and phase/delay.
- Magnitude approximation
  (If phase or delay performance is important , allpass filter can be used to achieve the necessary phase correction.
  1. Butterworth response : maximally flat magnitude in the passband.
  2. Chebyshev response : equal ripple in the passband.
  3. Elliptic response : equal ripple in both the passband and stopband.(Low order and the most economical realization)

Approximation Methods(Cont.)

4. Gaussian response :
   (a) freedom from ringing or overshoot.
   (b) symmetry about the time for which the response is a maximum.
5. And many others

- Phase or delay approximation
  Bessel-Thomson response : maximally flat delay.

Example

- Butterworth $|H(jw)|^2 = \dfrac{1}{1+\varepsilon^2\omega^{2n}}$

Assume ε=1 $|H(s)|^2 = \dfrac{1}{1+\omega^{2n}} = \dfrac{1}{1+(\frac{s}{j\Omega_0})^{2n}}$

Assume $\Omega_0 = 1 (\text{i.e.normalized})$

$$|H(s)|^2 = \frac{1}{1-S^{2n}}, \ |H(s)|^2 = H(S)H(-S) = \frac{1}{1-S^{2n}}$$

1. For n=1 $H(S)H(-S) = \dfrac{1}{1-S^2} = (\dfrac{1}{1+s})(\dfrac{1}{1-S})$

=> $H(S) = (\dfrac{1}{1+s})$ (Only the left-plane pole is selected)

The denominator of realizable filter function must be Hurwitz polynomials

Example(Cont.)

2. For n=2

$$H(S)H(-S)=\frac{1}{[S+\frac{1}{\sqrt{2}}+\frac{j}{\sqrt{2}}][S-\frac{1}{\sqrt{2}}+\frac{j}{\sqrt{2}}][S+\frac{1}{\sqrt{2}}-\frac{j}{\sqrt{2}}][S-\frac{1}{\sqrt{2}}-\frac{j}{\sqrt{2}}]}$$

$$=> \quad H(S)H(-S)=\frac{1}{[S+\frac{1}{\sqrt{2}}+\frac{j}{\sqrt{2}}][S+\frac{1}{\sqrt{2}}-\frac{j}{\sqrt{2}}]}$$

- Similarly transfer functions of other types can be derived.
- Refer to appendix III

Frequency Transformations

- Lowpass prototype => $\left\{ \begin{array}{l} \text{highpass} \\ \text{bandpass} \\ \text{bandreject} \\ \cdots\cdots \end{array} \right.$
  - Lowpass prototype $\overline{S} = \overline{a} + j\overline{w}$
    frequency variable $\overline{s}$ which is normalized such that $\overline{w} = 1$ at the edge of the lowpass passband
  - Target filter S=a+jw
  - Frequency transformation $\overline{s} = F(s)$ such that

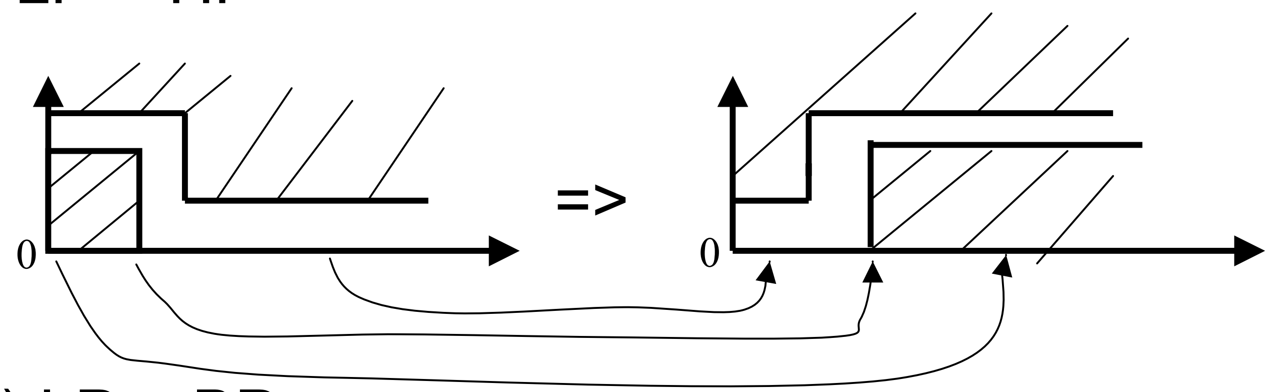
$0 \leq |\overline{w}| \leq 1 \xrightarrow{\text{mapped to}}$ passband of target filter

$|\overline{w}| > 1 \xrightarrow{\text{mapped to}}$ stopband of target filter

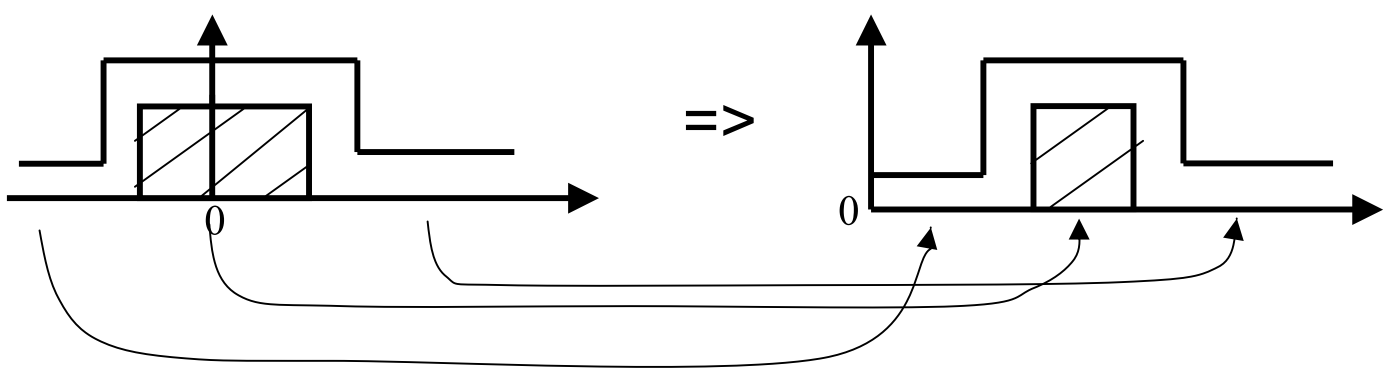
Frequency Transformations(Cont.)

- Examples

(i) LP=>HP



(ii) LP=>BP

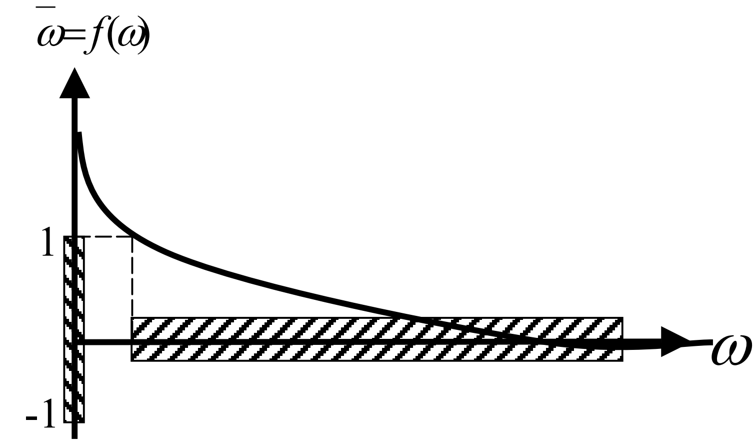


(iii) LP=>two-passband BP

Frequency Transformations(Cont.)

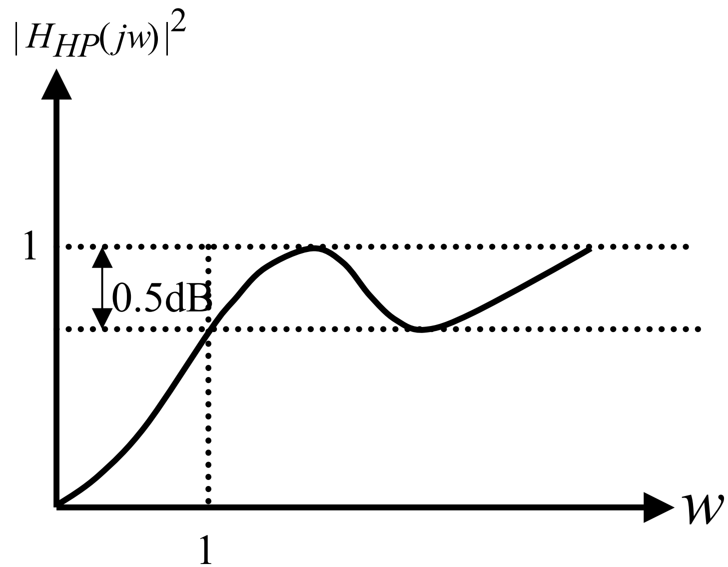
- Lowpass to highpass

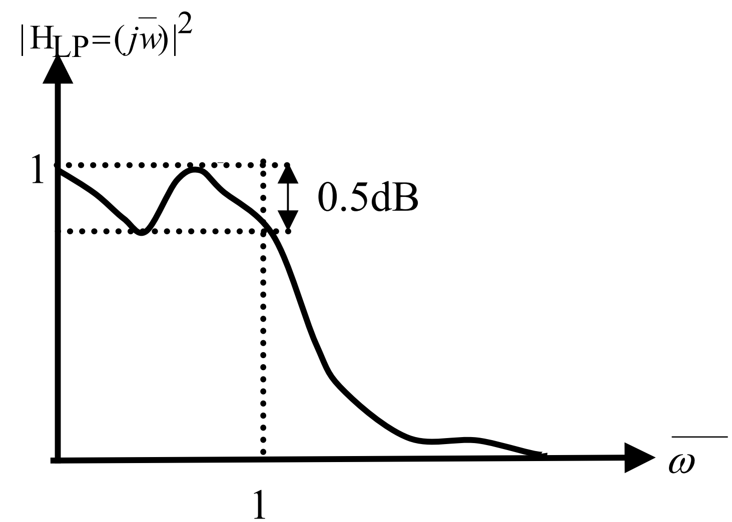
(i) $\bar{s} = \dfrac{1}{s} \xrightarrow{s=j\omega} \overline{w} = \dfrac{1}{w}$



$$e.g.\ H_{LP}(\bar{s}) = \frac{0.716}{\bar{s}^3 + 1.253\bar{s}^2 + 1.535\bar{s} + 0.716}$$

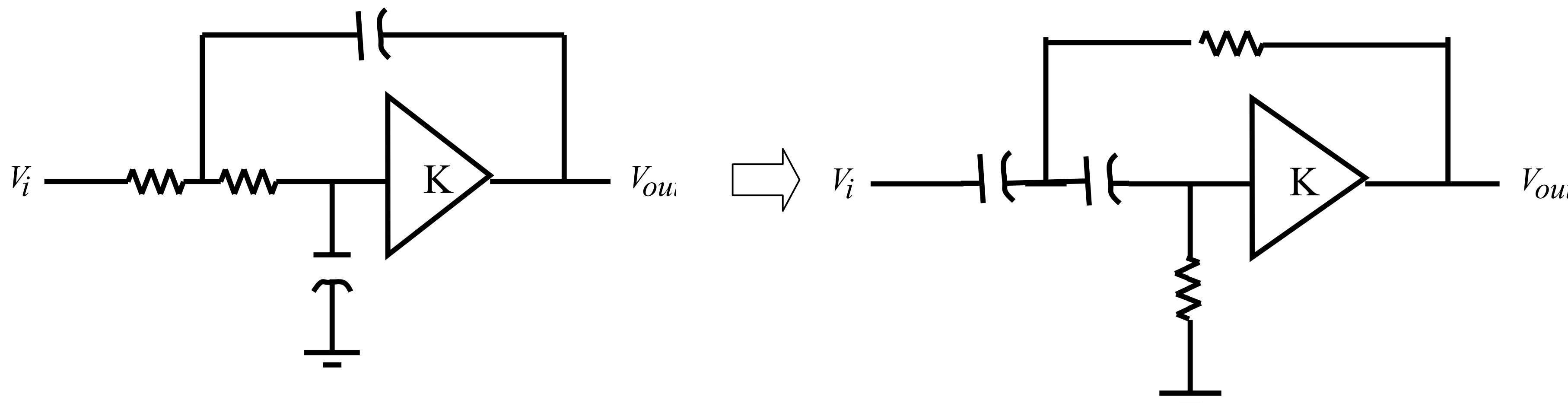
$$\Rightarrow H_{HP} = \frac{0.716S^3}{0.716S^3 + 1.535S^2 + 1.253S + 1}$$





Frequency Transformations(Cont.)

(ii) RC : CR transformation



$$H_{LP}(s) = \frac{bH(0)}{s^2 + as + b}$$

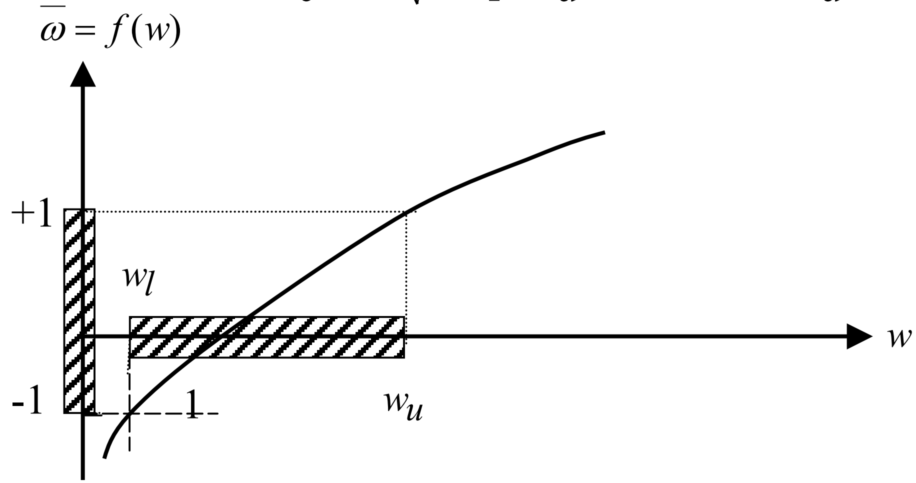
$$H_{LP}(s) = \frac{H(\infty)S^2}{S^2 + aS + b}$$

Frequency Transformations(Cont.)

- LP=>BP

$$\overline{S} = \frac{\Omega_0}{B}\frac{S^2+1}{S} = Q\frac{S^2+1}{S} \xrightarrow{s=j\omega} \overline{w} = Q\frac{\omega^2-1}{\omega}$$

where $\Omega_0 = \sqrt{\Omega_l \Omega_u}$ & $B = \Omega_u - \Omega_l$



  - Example 1-14

- LP=>BR

$$\overline{S} = \frac{1}{Q}\frac{S}{S^2+1} \xrightarrow{S=jw} \overline{\omega} = -\frac{1}{Q}\frac{\omega}{\omega^2+1}$$

  - Example 1-15