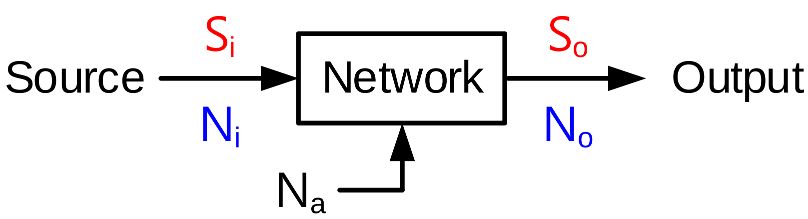
Filters

- The major reasons why filters are used
  - Attenuate out-of-band power
    - Avoid interference
    - Allow large in-band signal
    - Reduce signal slew-rate
  - Adjust in-band gain-phase relationship

- Electronic filters are important building blocks of communication and instrumentation systems.
- Filter design is one of the very few areas of engineering for which a complete design theory exists, starting from specification and ending with a circuit realization.

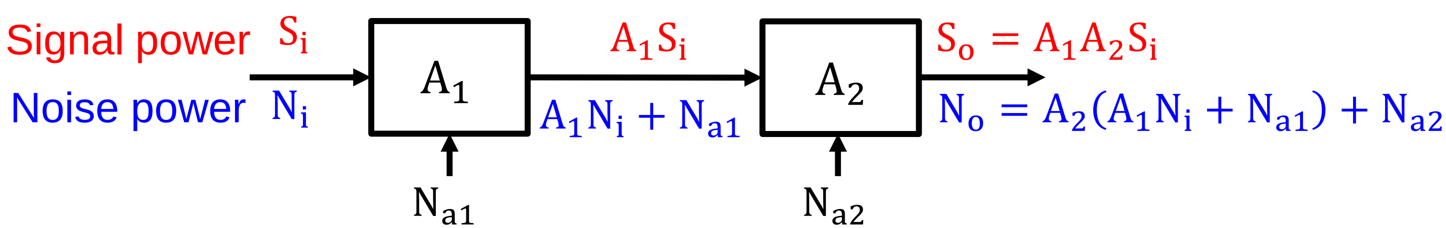
Noise Figure

- Definition of noise figure (NF)
  - The ratio of the signal-to-noise power ratio at the input to the signal-to-noise power ratio at the output



S_i : Signal power at the input
S_o : Signal power at the output
N_i : Noise power at the input
N_o: Noise power at the output
N_a: Additional noise power from the network

  - Example

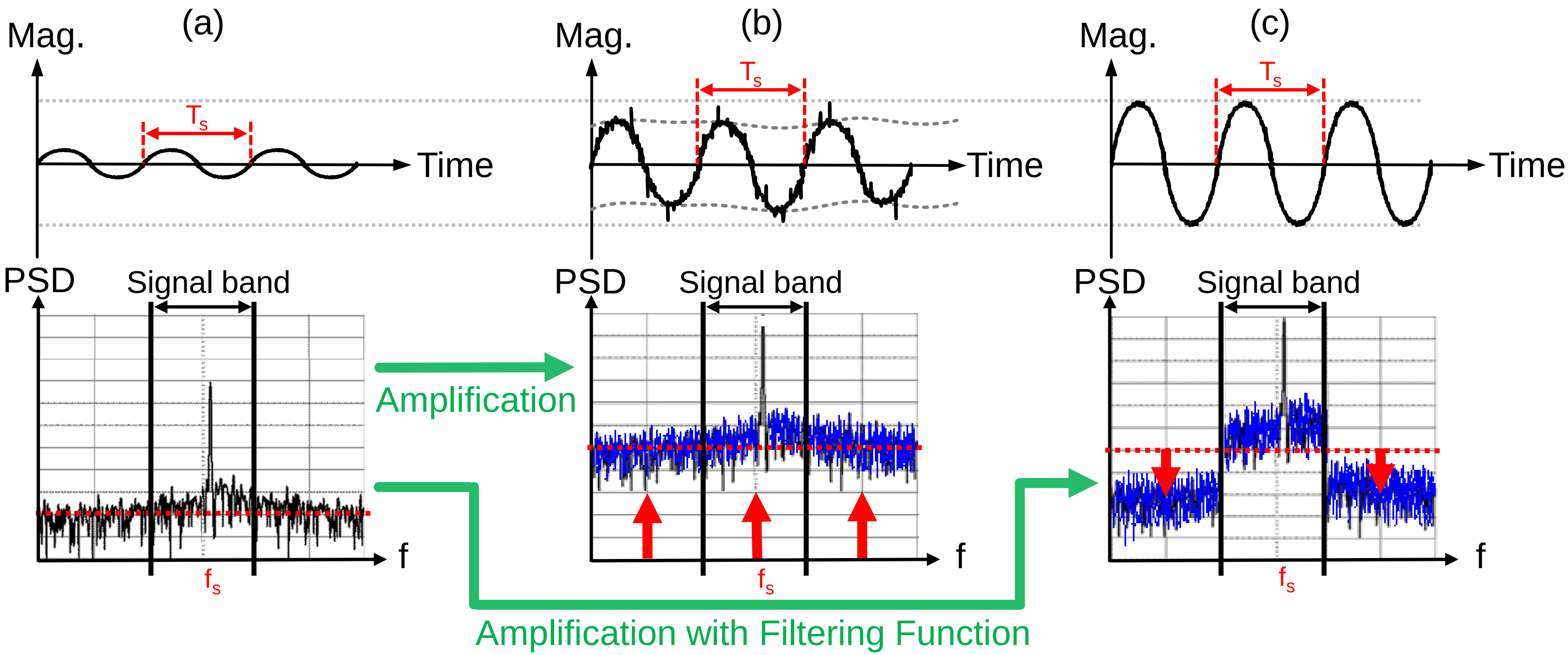


$$\text{Noise Figure (NF)} = \frac{S_i/N_i}{S_o/N_o} = \frac{S_i/N_i}{A_1 A_2 S_i/[A_2(A_1 N_i + N_{a1}) + N_{a2}]} = \frac{N_i + \frac{N_{a1}}{A_1} + \frac{N_{a2}}{A_1 A_2}}{N_i}$$

- Ideal NF = 1
- Large NF caused by noisy network
  - Low-noise amplifier (LNA): Higher A_1 with lower N_{a1} → Smaller NF

Noise Figure of Amplifier with Filtering Function

- Illustration in time domain and frequency domain



- Signal band: $\mathrm{NF}_{(a)} < \mathrm{NF}_{(b)}$ & $\mathrm{NF}_{(c)}$
- Function of filter: Attenuate out-of-band power
  - Allow larger in-band signal
  - Reduce signal slew rate
  - Avoid interference

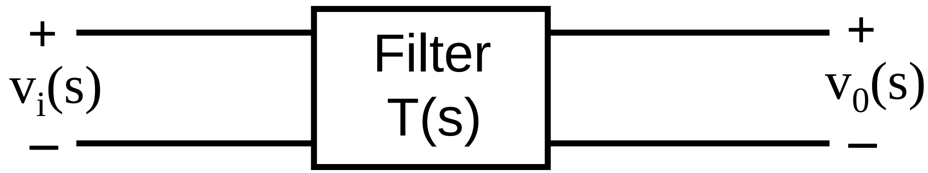
Mag. : Magnitude
PSD : Power spectral density
f : Frequency
$f_s = 1/T_s$

Filter Types(Implementation)

- Passive LC filters
  - Oldest technology
  - Work well at high frequency
  - For low frequency application (DC~100kHz), L is large and impossible to fabricate in IC

- Inductorless filters
  - Active-RC filters
  - Switched-capacitor filters

Filter Transfer Function



$$T(s)=\frac{V_o(s)}{V_i(s)} \qquad T(s)=|T(j\omega)|e^{j\theta(\omega)}$$

Gain $G(\omega)=20\log|T(j\omega)|$ Unit:dB

Attenuation $A(\omega)=20\,log\left|\frac{1}{T(j\omega)}\right|=-20\,log|T(j\omega)|=-G(j\omega)$ Unit:dB

$|V_o(j\omega)|=|T(j\omega)||V_i(j\omega)|$ where T(s) can be written as z

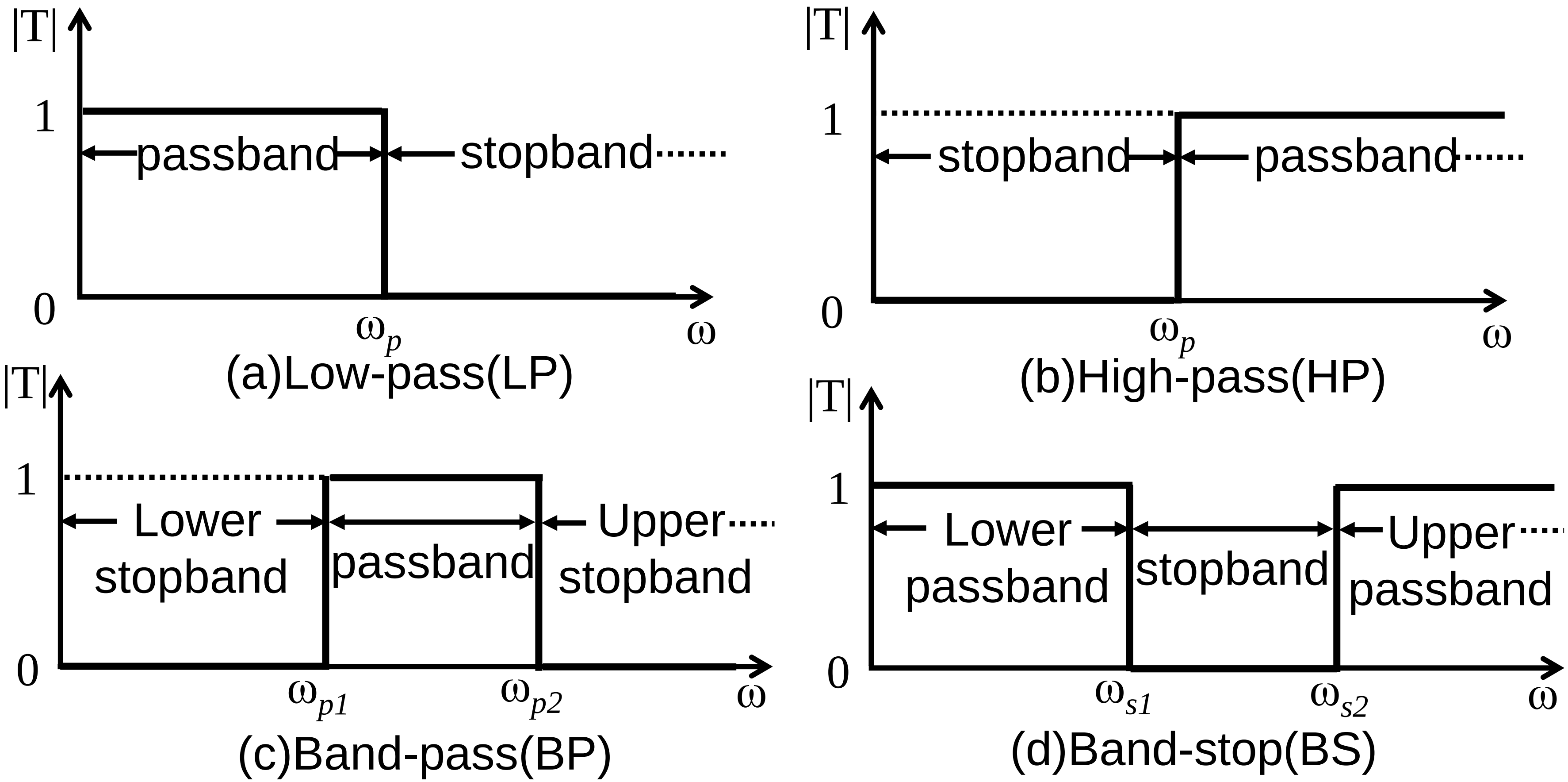
$$T(s)=\frac{a_M s^M + a_{M-1}s^{M-1}+\ldots+a_0}{s^N+b_{N-1}s^{N-1}+\ldots+b_0}=\frac{a_M(s-z_1)(s-z_2)\ldots(s-z_M)}{(s-p_1)(s-p_2)\ldots(s-p_N)}$$

where N is the filter order, $z_1, z_2, \ldots, z_M$ are zeros, and $p_1, p_2, .., p_N$ are poles

- N ≥ M for a realizable filter
- Poles and zeros must be real or complex conjugate pairs.
- To obtain zero or small stopband transmission, zeros are usually placed on the jω-axis at stopband frequencies

Filter Types (Characteristics)

- 4 major types : low-pass, high-pass, band-pass, and band-reject (or band-stop)
- Example: Brick-wall(ideal) response



Transmission Specifications

- Low-pass filter

  Low-pass is specified by four parameters

  ω_p passband edge
  ω_s stopband edge
  A_{max} passband ripple
  A_{min} stopband attenuation

  Ideal low-pass
  1. lower A_{max}
  2. higher A_{min}
  3. selectivity ratio
     $\omega_s / \omega_p \rightarrow 1$

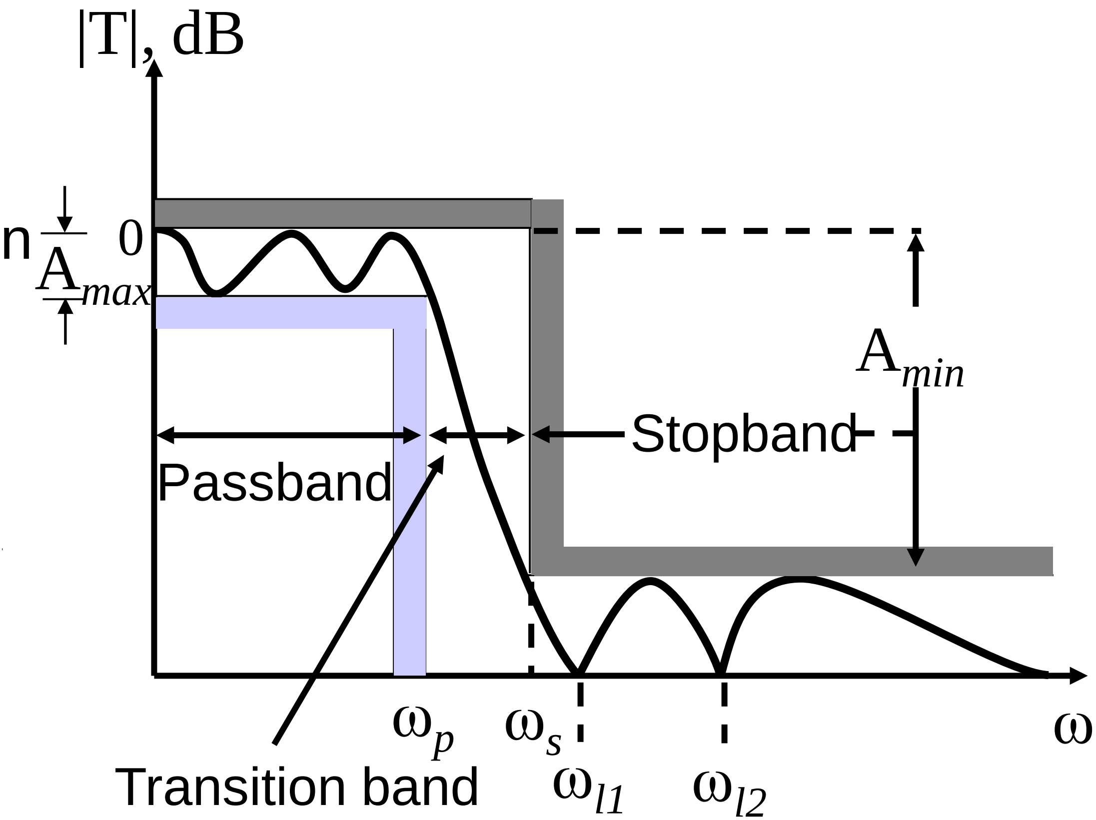


Fig. 11.3

Transmission Specifications (Cont.)

- Band-pass filter

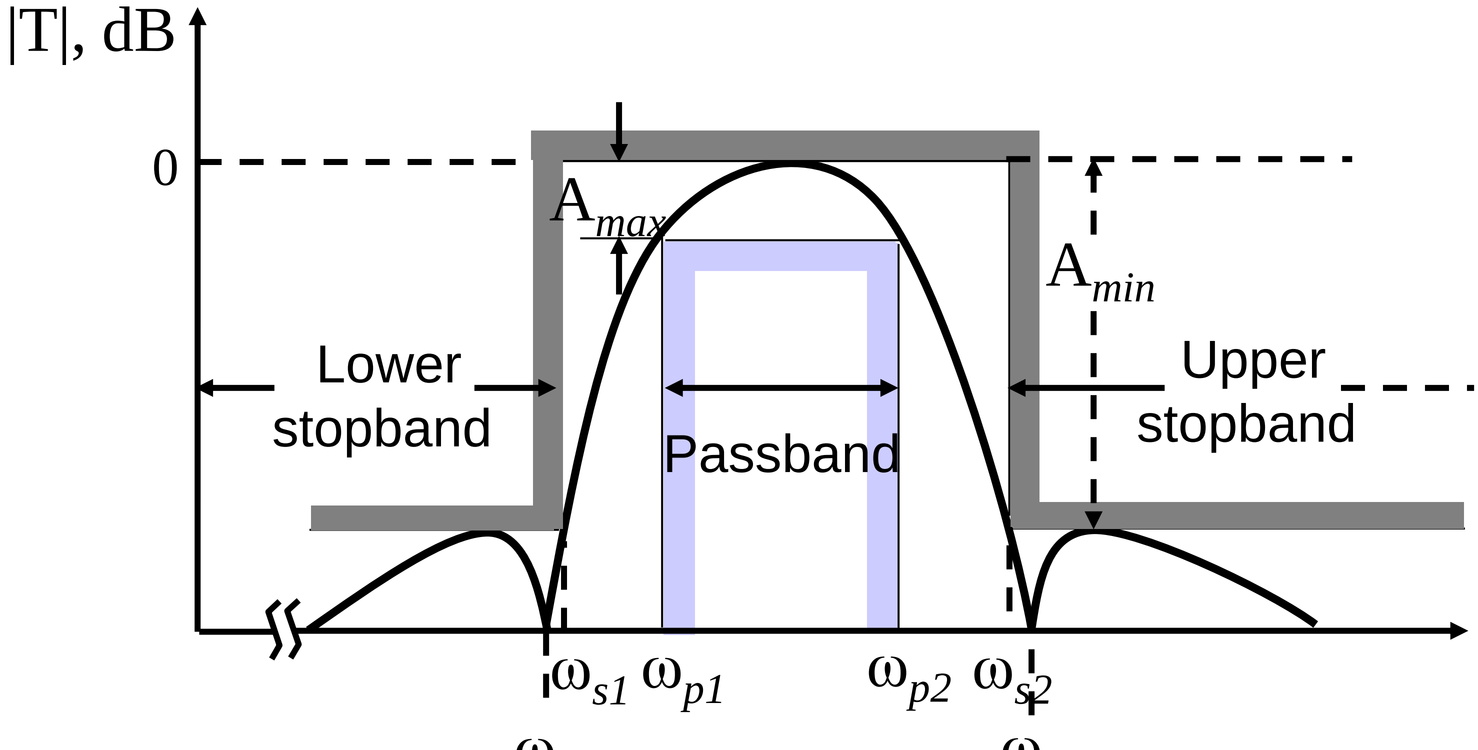


Fig. 11.4

- High-pass filter: similar terminology

Pole-Zero Pattern

- Assume a transfer function

$$T(s)=\frac{a_M s^M + a_{M-1}s^{M-1}+\ldots+a_0}{s^N+b_{N-1}s^{N-1}+\ldots+b_0}=\frac{a_M(s-z_1)(s-z_2)\ldots(s-z_M)}{(s-p_1)(s-p_2)\ldots(s-p_N)}$$

  - The magnitude, M, of T(s) is

$$M=\frac{a_M\cdot|s-z_1|\cdot|s-z_2|\cdot\ldots\cdot|s-z_M|}{|s-p_1|\cdot|s-p_2|\cdot\ldots\cdot|s-p_N|}=a_M\cdot\frac{\prod_{i=1}^{M}\text{distance between } s \text{ and } z_i}{\prod_{i=1}^{N}\text{distance between } s \text{ and } p_i}$$

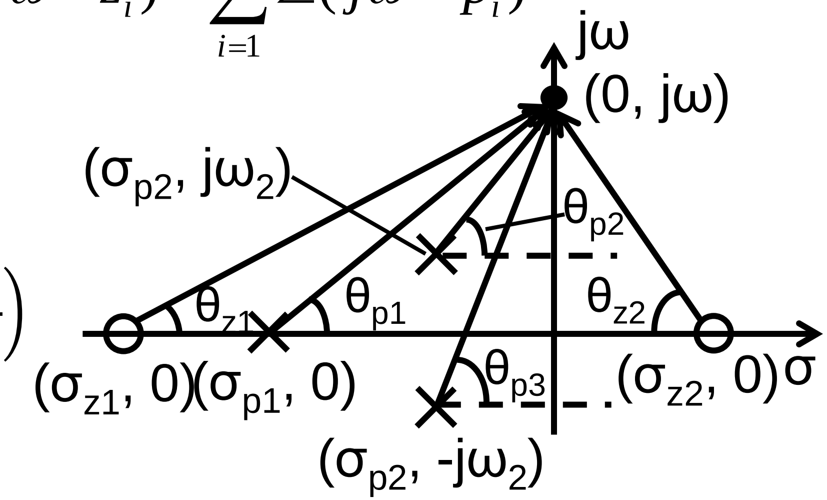
  - The angle, θ, of T(s) is $\theta=\sum_{i=1}^{M}\angle(j\omega-z_i)-\sum_{i=1}^{N}\angle(j\omega-p_i)$

    e.g. three-pole two-zero system

$$\Rightarrow \theta=\theta_{z1}-\theta_{z2}-\theta_{p1}-\theta_{p2}-\theta_{p3}$$

$$=\tan^{-1}\left(\frac{\omega}{\sigma_{z1}}\right)-\tan^{-1}\left(\frac{\omega}{\sigma_{z2}}\right)-\tan^{-1}\left(\frac{\omega}{\sigma_{p1}}\right)$$

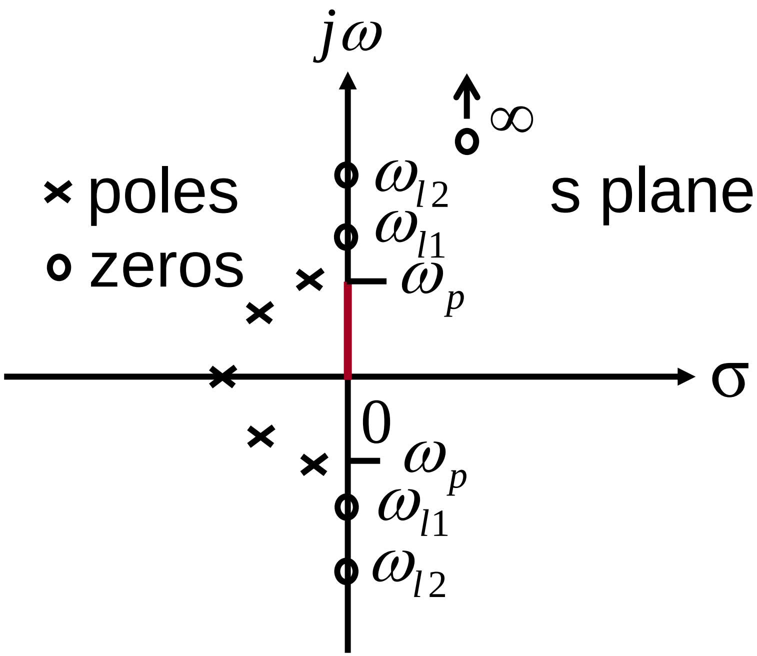
$$-\tan^{-1}\left(\frac{\omega-\omega_2}{\sigma_{p2}}\right)-\tan^{-1}\left(\frac{\omega+\omega_2}{\sigma_{p2}}\right)$$



- For a given pole-zero pattern, you can sketch its transfer function by hand calculation

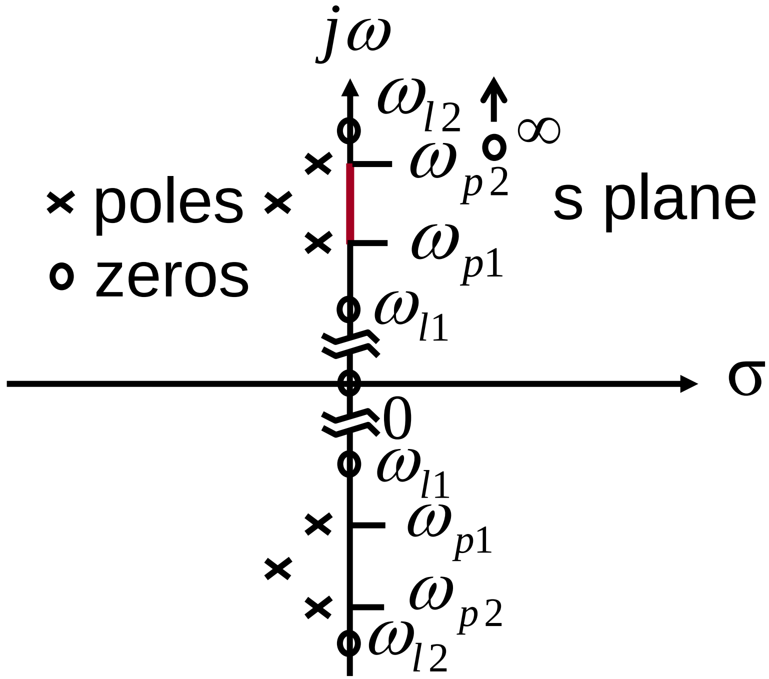
Examples

- Pole-zero pattern for a low-pass filter (N=5) as in Fig. 11.3 (P.13-7)



$$T(s)=\frac{a_4(s^2+\omega_{l1}{}^2)(s^2+\omega_{l2}{}^2)}{s^5+b_4s^4+b_3s^3+b_2s^2+b_1s+b_0}$$

- Pole-zero pattern for a band-pass filter (N=6) as in Fig. 11.4 (P.13-8)

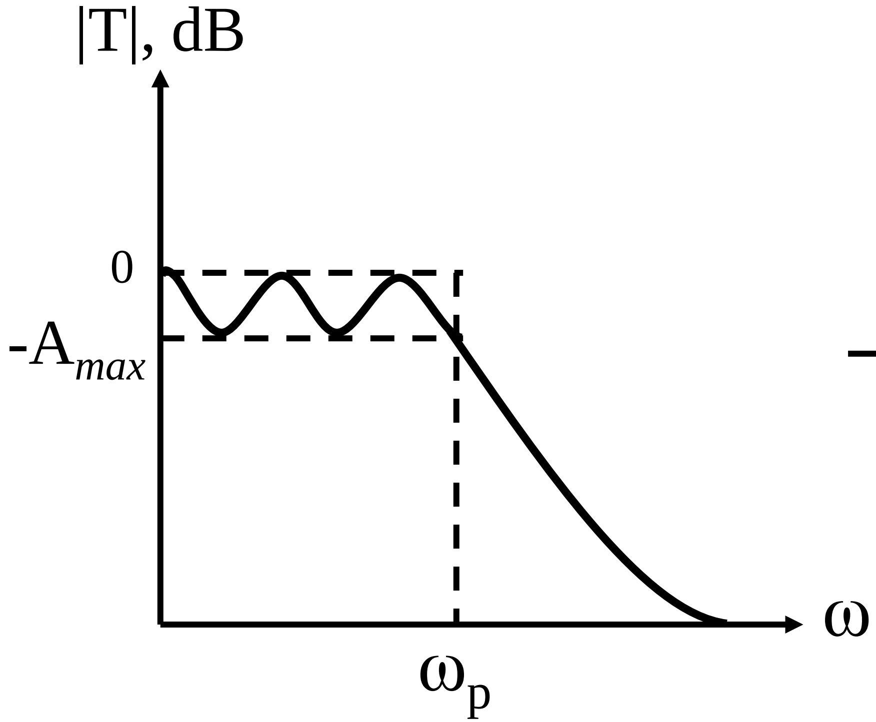


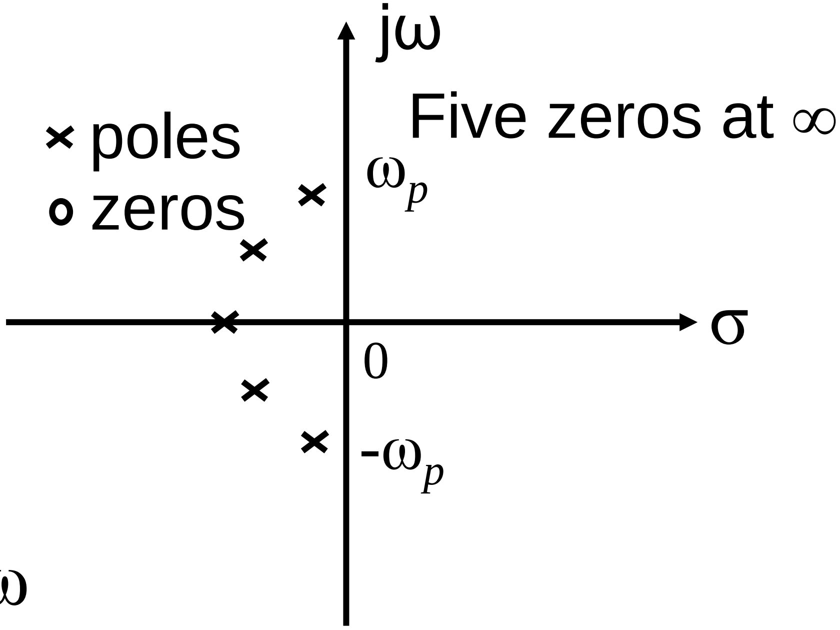
$$T(s)=\frac{a_4s(s^2+\omega_{l1}{}^2)(s^2+\omega_{l2}{}^2)}{s^6+b_5s^5+b_4s^4+b_3s^3+b_2s^2+b_1s+b_0}$$

Examples (Cont.)

- All-pole filter
  - Typical example (5th-order)

$$T(s)=\frac{a_M}{s^N+b_{N-1}s^{N-1}+\ldots+b_0}$$



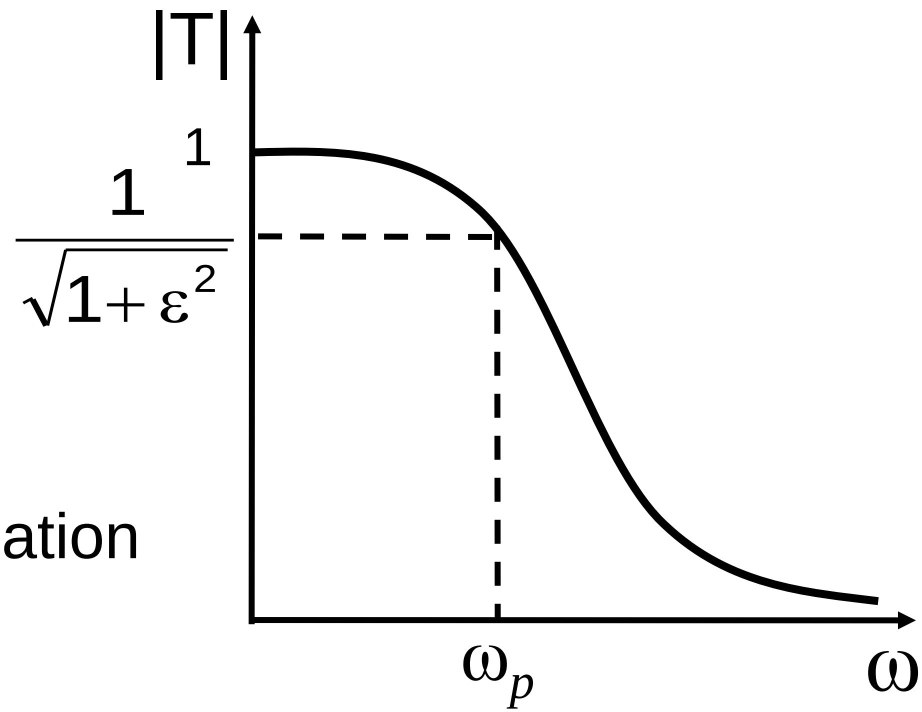


Butterworth Filters

- All-pole filter
- N is filter order and ω_p is passband edge
  - $|T(j\omega)| = \dfrac{1}{\sqrt{1+\varepsilon^2(\omega/\omega_p)^{2N}}}$
  - At $\omega = \omega_p$;

    $|T(j\omega_p)| = \dfrac{1}{\sqrt{1+\varepsilon^2}}$
  - Maximum passband variation

    $A_{max} = 20\log\sqrt{1+\varepsilon^2}$



Butterworth Filters (Cont.)

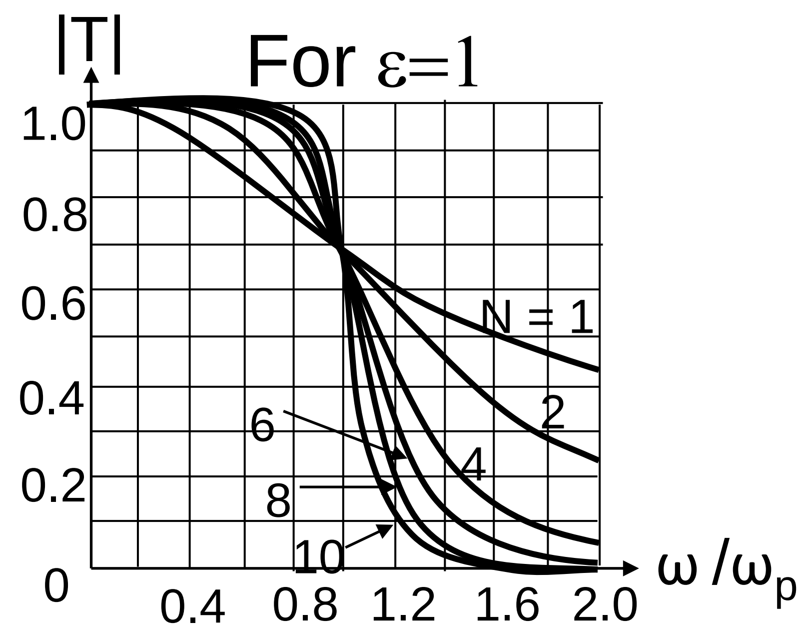
- Maximally flat response
  - First 2N-1 derivative of |T| relative to ω are zero at $\omega=0$

  ⇨ Response is very flat near $\omega=0$
  - Order N ↑ ⇨ passband flatness ↑
- Attenuation at stopband edge $\omega=\omega_s$

$$A(\omega_s) = -20\log\frac{1}{\sqrt{1+\varepsilon^2(\omega_s/\omega_p)^{2N}}}$$

$$=10\log\left(1+\varepsilon^2(\omega_s/\omega_p)^{2N}\right)$$

Filter order can usually be obtained from $A(\omega_s)$



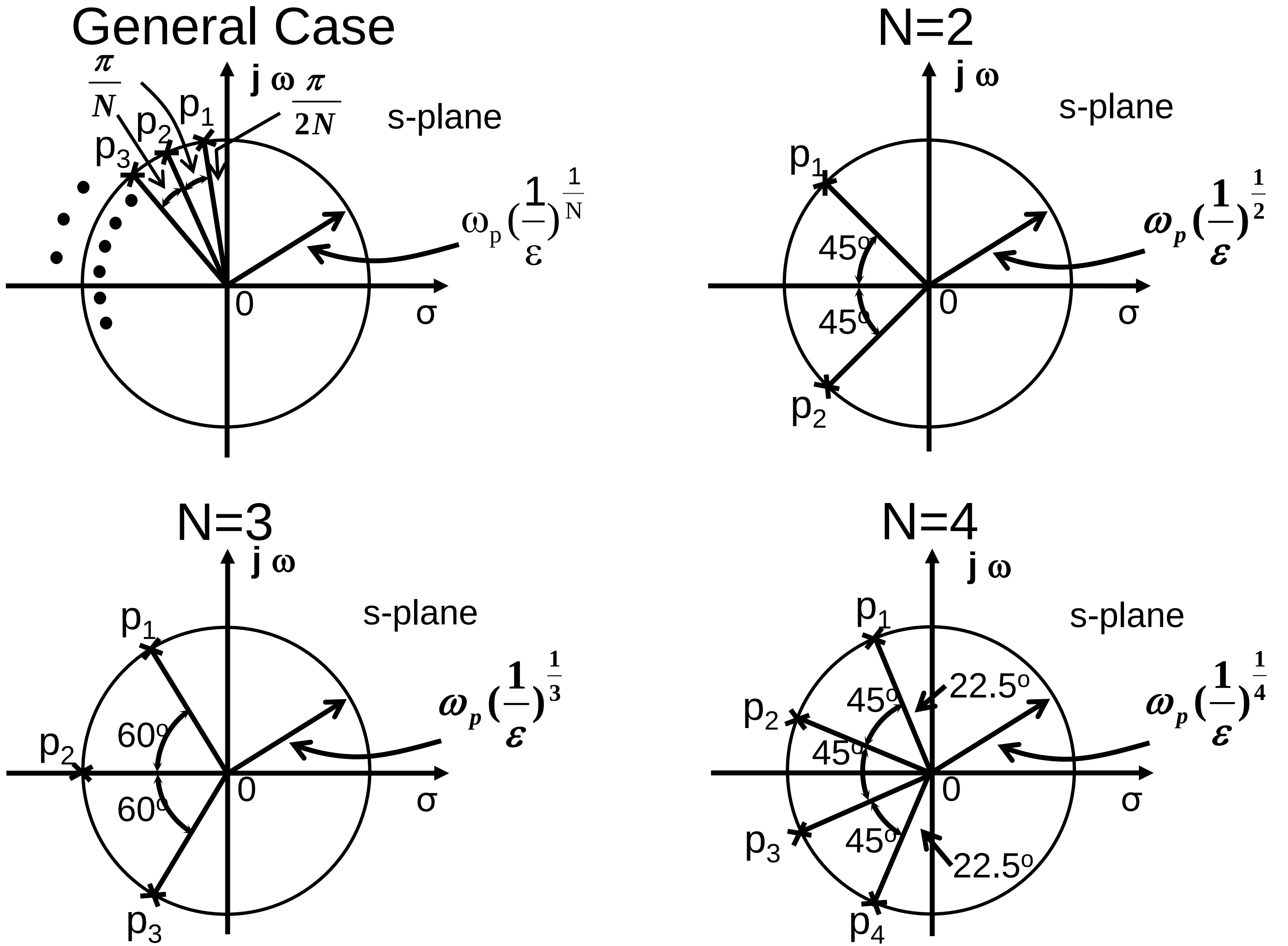
Butterworth Filters (Cont.)

- Butterworth filter design methodology to meet specification
  1. Determine ε from A_{max}
  2. Determine N, the number of order, by $A(\omega_s) > A_{min}$
  3. Determine poles using the graphical construction shown in the next page
  4. Transfer function

$$T(s) = \frac{K{\omega_0}^N}{(s-p_1)(s-p_2)...(s-p_N)}$$

$$\text{where K is DC gain, } \omega_0 = \omega_p (\frac{1}{\varepsilon})^{\frac{1}{N}}$$

Poles of Butterworth Filter



Normalized Butterworth Polynomials

- $T(s) \xrightarrow{S=j\omega} T(j\omega)$

$$T(j\omega)T(-j\omega) = \left|T(j\omega)\right|^2 = \frac{1}{1+\varepsilon^2(\frac{\omega}{\omega_p})^{2N}} = \frac{1}{1+(-1)^N\varepsilon^2(\frac{j\omega}{\omega_p})^{2N}}$$

$$\left|T(s)\right|^2 = \frac{1}{1+(-1)^N\varepsilon^2(\frac{s}{\omega_p})^{2N}}$$

poles: $\varepsilon^2(\frac{s}{\omega_p})^{2N} = -(-1)^N \Rightarrow (\frac{s}{\omega_p})^{2N} = -\frac{(-1)^N}{\varepsilon^2}$

$$\Rightarrow \begin{cases} \dfrac{-1}{\varepsilon^2} = \dfrac{e^{j(\pi+\ell 2\pi)}}{\varepsilon^2}; & \text{N is even} \\ \dfrac{1}{\varepsilon^2} = \dfrac{e^{j\ell 2\pi}}{\varepsilon^2}; & \text{N is odd} \end{cases}$$

where $\ell = 0, 1, 2, ..., 2N-1$

Normalized Butterworth Polynomials (Cont.)

- e.g. N=2 & ε=1

$$(\frac{s}{\omega_p})^4 = -1 = e^{j(\pi + \ell 2\pi)}$$

$$s = \omega_p e^{j(\frac{\pi}{4} + \frac{\ell}{4} 2\pi)} \ ; \ell = 0,1,2,3$$

poles:

$$\begin{cases} s_1 = \omega_p e^{j\frac{\pi}{4}} = \omega_p(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}) \\ s_2 = \omega_p e^{j(\frac{\pi}{4}+\frac{\pi}{2})} = \omega_p(-\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}) \\ s_3 = \omega_p e^{j(\frac{\pi}{4}+\pi)} = \omega_p(-\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}) \\ s_4 = \omega_p e^{j(\frac{\pi}{4}+\frac{3\pi}{2})} = \omega_p(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}) \end{cases}$$

Normalized Butterworth Polynomials (Cont.)

- Normalized polynomials for N=2
  1. Take left plane poles
  2. Let ω_p =1 $\longrightarrow$ frequency normalization

$$T(s) = \frac{1}{(1-\frac{s}{s_2})(1-\frac{s}{s_3})} = \frac{1}{1+\sqrt{2}s+s^2} \quad \text{for} \quad N=2$$

$$B_2(s) = 1+\sqrt{2}s+s^2$$

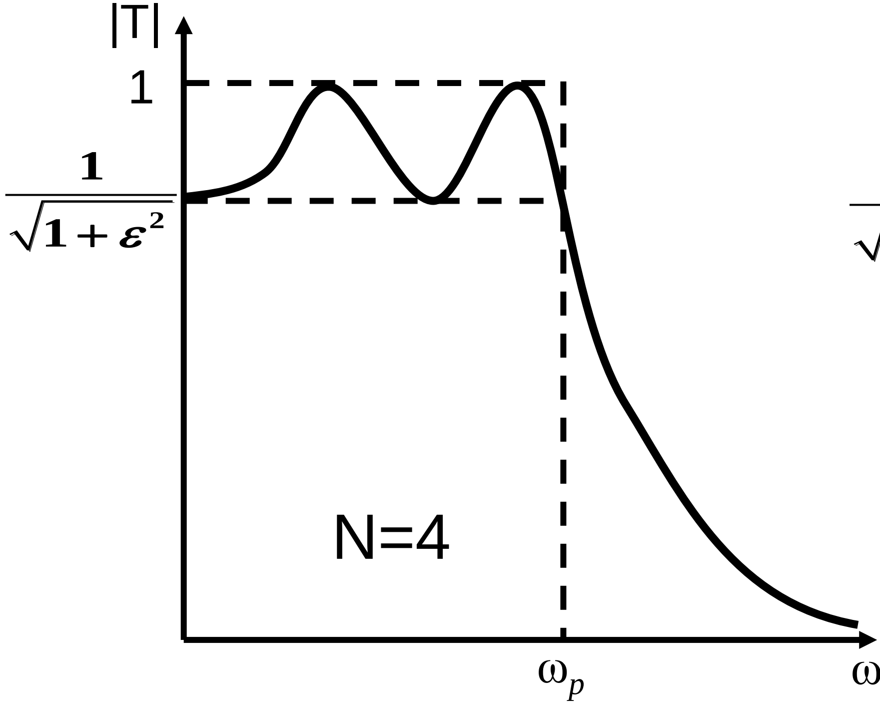
- Table of normalized Butterworth polynomials

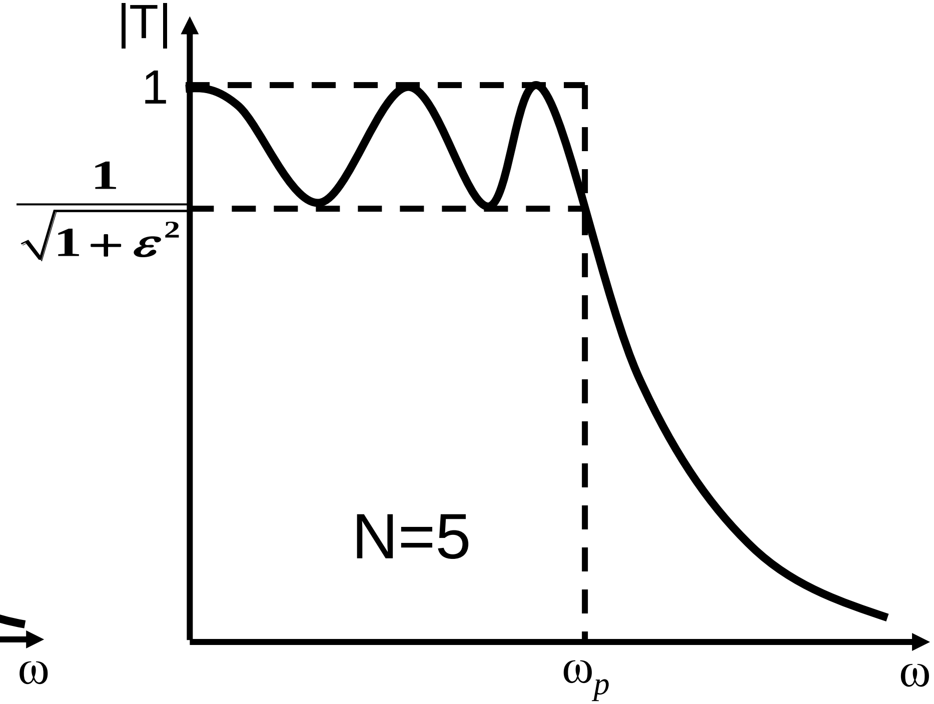
n	Factors of polynomial $B_n(s)$
1	$(s+1)$
2	$(s^2+1.414s+1)$
3	$(s+1)(s^2+s+1)$
4	$(s^2+0.765s+1)(s^2+1.848s+1)$
5	$(s+1)(s^2+0.618s+1)(s^2+1.618s+1)$
6	$(s^2+0.518s+1)(s^2+1.414s+1)(s^2+1.932s+1)$
7	$(s+1)(s^2+0.445s+1)(s^2+1.247s+1)(s^2+1.802s+1)$
8	$(s^2+0.390s+1)(s^2+1.111s+1)(s^2+1.663s+1)(s^2+1.962s+1)$

Chebyshev Filters

- Equiripple in the passband
- Monotonically decreasing in the stopband
- Odd-order filter exhibits $|T(0)| = 1$

  Even-order filter exhibits $|T(0)| = |T(\omega_p)| = A_{max}$





Chebyshev Filters (Cont.)

- Transfer function

$$\left|T(j\omega)\right| = \frac{1}{\sqrt{1+\varepsilon^2\cos^2\left[N\cos^{-1}\left(\omega/\omega_p\right)\right]}} \qquad \text{for } \omega \le \omega_p$$

$$\left|T(j\omega)\right| = \frac{1}{\sqrt{1+\varepsilon^2\cosh^2\left[N\cosh^{-1}\left(\omega/\omega_p\right)\right]}} \qquad \text{for } \omega > \omega_p$$

$$\left|T(j\omega)\right| = \frac{1}{\sqrt{1+\varepsilon^2}}, \quad \text{when } \omega = \omega_p$$

$$A_{\max} = 10\log\left(1+\varepsilon^2\right)$$

$$Given\ A_{\max}, \varepsilon \text{ is determined}, \varepsilon = \sqrt{10^{A_{\max}/10} - 1}$$

- Attenuation at stopband edge

$$A(\omega_s) = 10\log\left[1+\varepsilon^2\cosh^2\left(N\cosh^{-1}\frac{\omega_s}{\omega_p}\right)\right]$$

Chebyshev Filters (Cont.)

- Chebyshev filter design methodology to meet specification
  1. Determine ε from A_{max}
  2. Determine N, the number of order, by $A(\omega_s) > A_{min}$
  3. Determine poles using Chebyshev equation

$$p_K = -\omega_p \sin\left(\frac{2k-1}{N}\frac{\pi}{2}\right)\sinh\left(\frac{1}{N}\sinh^{-1}\frac{1}{\varepsilon}\right) + j\omega_p \cos\left(\frac{2k-1}{N}\frac{\pi}{2}\right)\cosh\left(\frac{1}{N}\sinh^{-1}\frac{1}{\varepsilon}\right)$$

  4. Transfer function

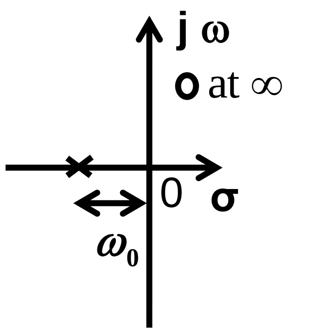
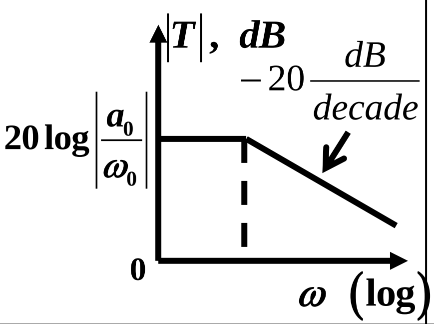
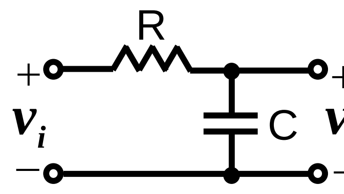
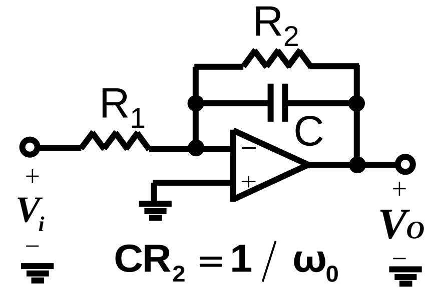
$$T(s) = \frac{K{\omega_p}^N}{\varepsilon \cdot 2^{N-1}(s-p_1)(s-p_2)\ldots(s-p_N)}$$

where K is DC gain

Reading Assignment: First-Order Filter Function

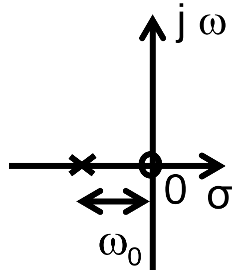
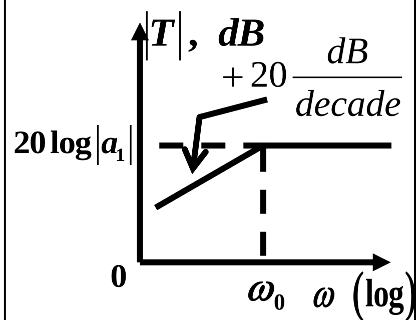
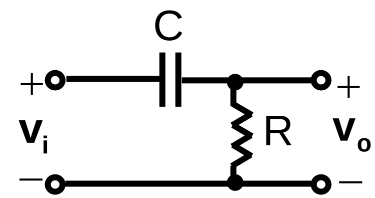
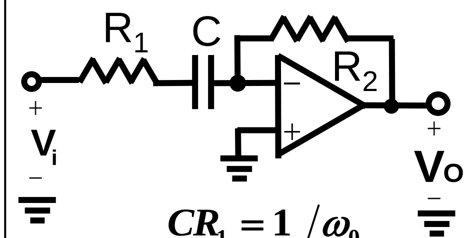
- First-order transfer function $T(s) = \dfrac{a_1 s + a_0}{s + \omega_0}$
- pole at $s = -\omega_0$
- Transmission zero at $s = -a_0/a_1$
- $T(j\infty) = a_{1:}$

(a) Low-pass (LP) $\Rightarrow T(s) = \dfrac{a_0}{s + \omega_0}$

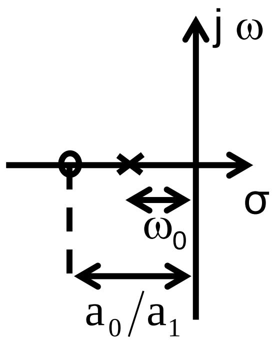
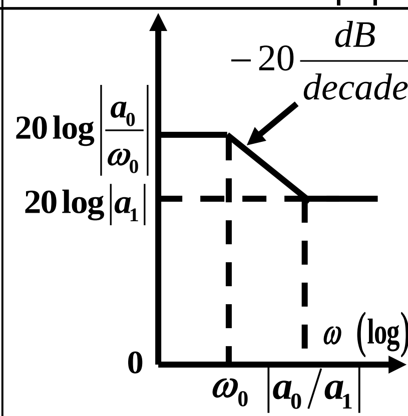
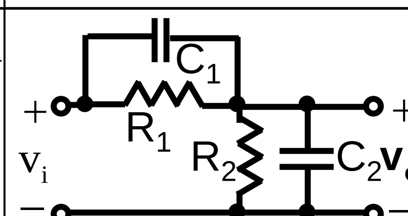
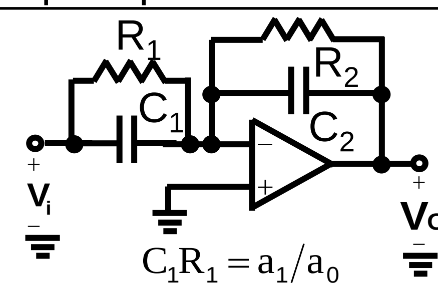
S-Plane poles/zeros	Bode Plot for |T|	Passive Realization	Op Amp-RC Realization
		 $CR = 1/\omega_0$ dc gain $= 1$	 $CR_2 = 1/\omega_0$ dc gain $= -R_2/R_1$

Reading Assignment: First-Order Filter Function (Cont.)

(b) High-pass (HP) $\Rightarrow T(s) = \dfrac{a_1 s}{s + \omega_0}$

S-Plane poles/zeros	Bode Plot for |T|	Passive Realization	Op Amp-RC Realization
		 $CR = 1/\omega_0$ **High – frequency gain = 1**	 $CR_1 = 1/\omega_0$ **High – frequency** gain $= -R_2/R_1$

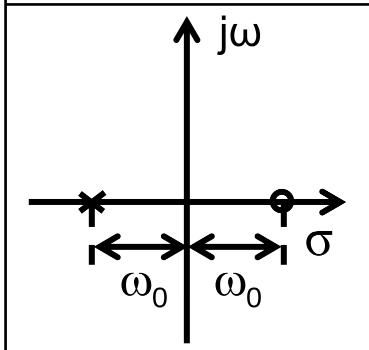
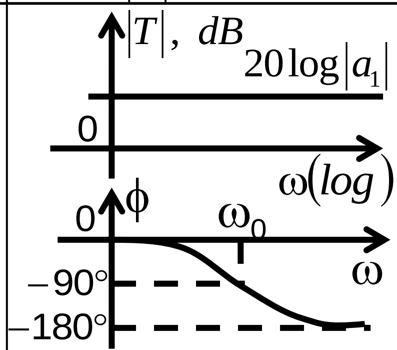
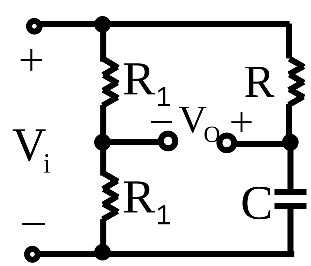
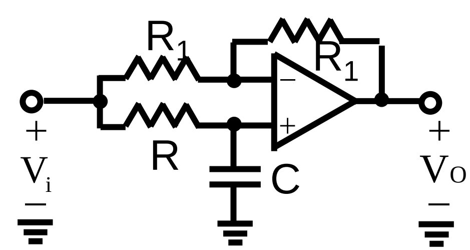
(c) General $\Rightarrow T(s) = \dfrac{a_1 s + a_0}{s + \omega_0}$

S-Plane poles/zeros	Bode Plot for |T|	Passive Realization	Op Amp-RC Realization
		 $(C_1 + C_2)(R_1 // R_2) = 1/\omega_0$ $C_1R_1 = a_1/a_0$ DC gain $= R_2/(R_1 + R_2)$ HF gain $= C_1/(C_1 + C_2)$	 $C_1R_1 = a_1/a_0$ $C_2R_2 = 1/\omega_0$ DC gain $= -R_2/R_1$ HF gain $= -C_1/C_2$

Reading Assignment: First-Order All-pass Filter

- Special case of first-order filter
- Transmission zero and pole are symmetrically located relative to jω–axis
- Transmission is constant at all frequencies
- phase is not constant at all frequencies
- Most often used in the design of delay equalizers

(d) All-pass (AP) $\Rightarrow T(s) = -a_1 \dfrac{s-\omega_0}{s+\omega_0} \quad a_1 > 0$

S-Plane poles/zeros	$\lvert T \rvert$ and ϕ	Passive Realization	Op Amp-RC Realization
		 $CR = 1/\omega_0$ Flat gain $(a_1) = 0.5$	 $CR = 1/\omega_0$ Flat gain $(a_1) = 1$

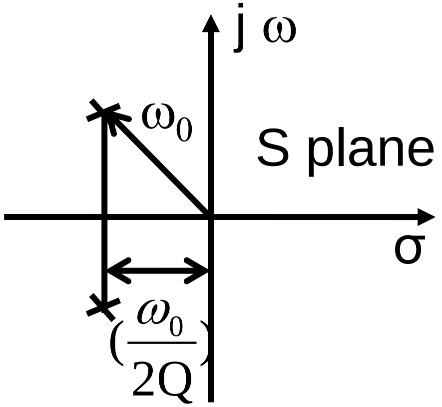
Reading Assignment: Second-Order Filter Function

- Second-order filter(also called biquadratic filter)

$$T(s) = \frac{a_2 s^2 + a_1 s + a_0}{s^2 + (\omega_0 / Q)s + \omega_0^2}$$

where ω_0 and Q determine poles according to

$$p_1,\ p_2 = -\frac{\omega_0}{2Q} \pm j\omega_0 \sqrt{1-(1/4Q^2)}$$

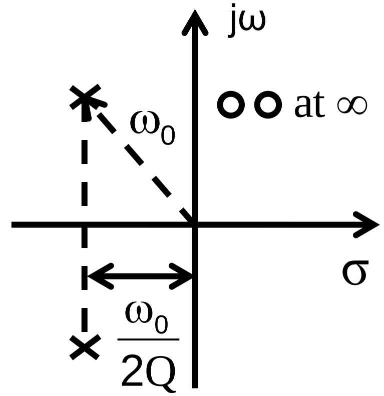
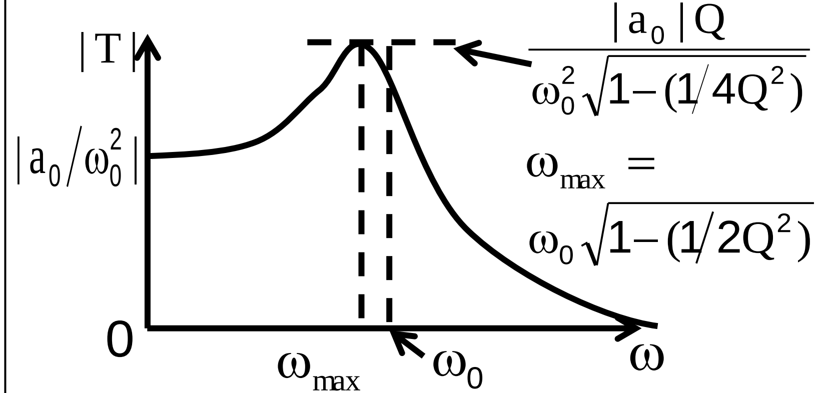
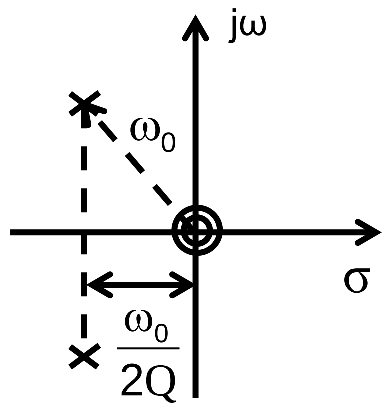
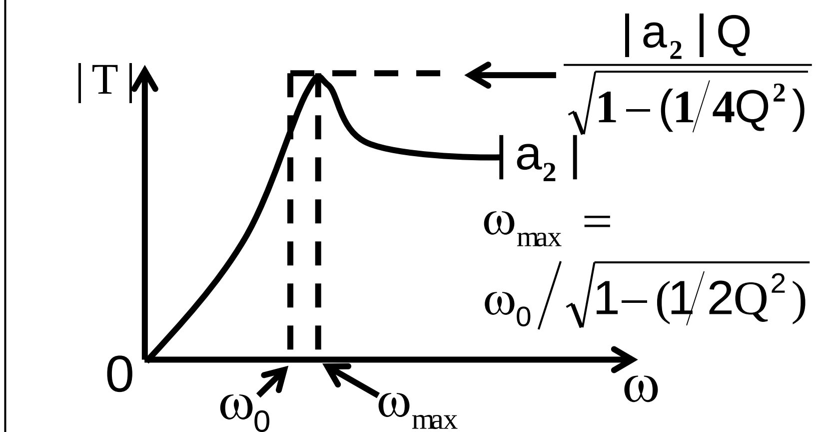


ω_0: distance of pole (from origin)

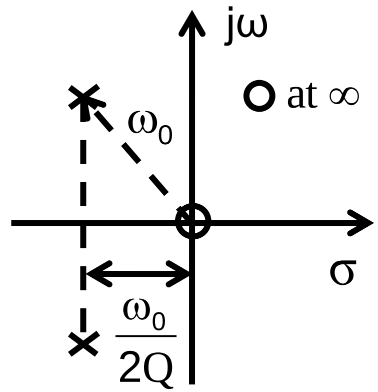
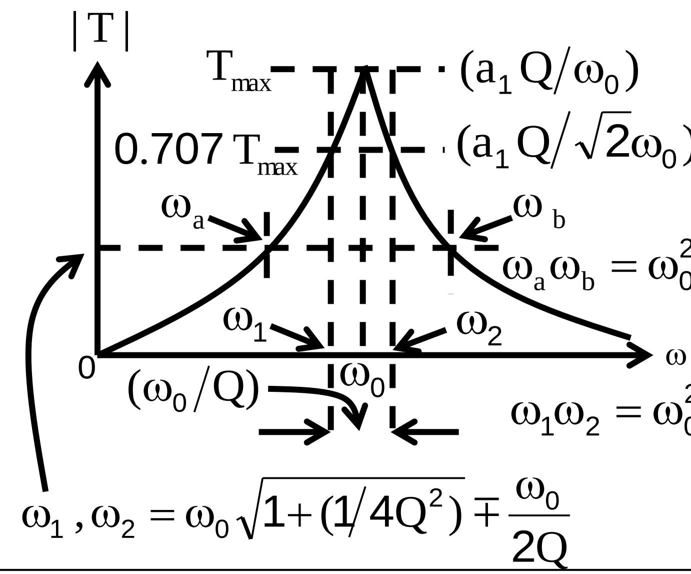
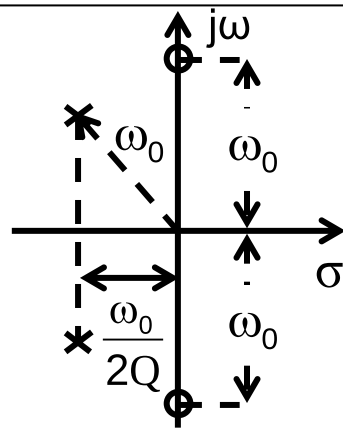
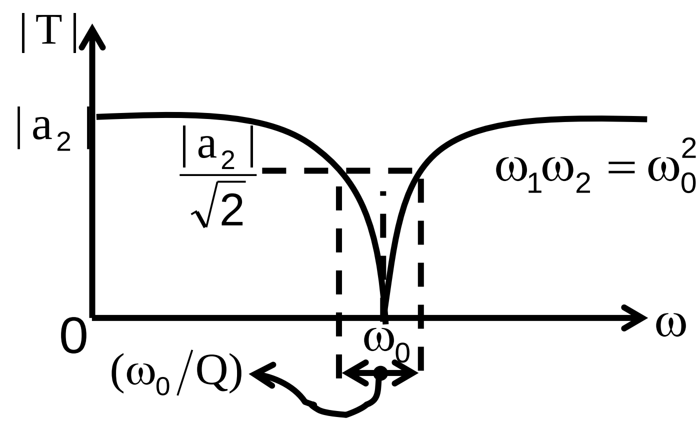
Q: pole quality factor

- ◆ Higher Q → higher selectivity(pole is closer to jω–axis)
- ◆ Q = ∞ → poles are on the jω–axis
  → Can yield sustained oscillation
- ◆ Q is negative → poles are in the right half of s-plane
  → Unstable

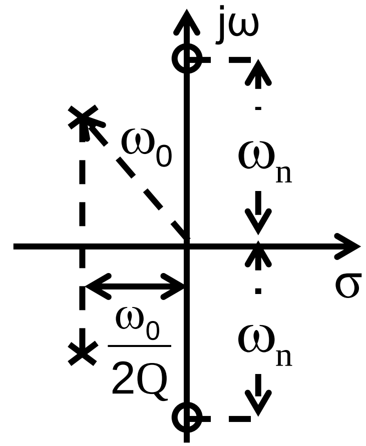
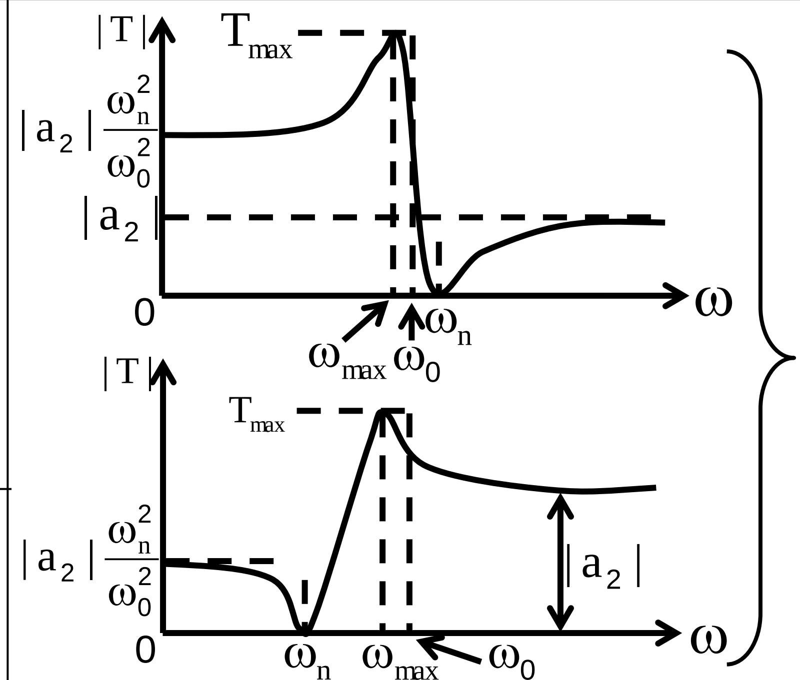
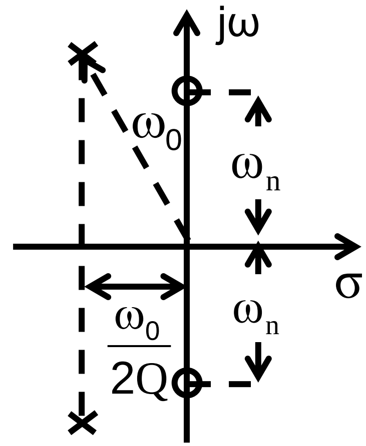
Reading Assignment: Second-Order Filter Function (Cont.)

<table>
<tr><th>Filter Type and T(s)</th><th>S-Plane poles/zeros</th><th>|T|</th></tr>
<tr><td>(a) Low-pass (LP)
$T(s)=\dfrac{a_0}{s^2+s(\omega_0/Q)+\omega_0^2}$
$\text{dc gain}=a_0/\omega_0^2$
$Q=1/\sqrt{2}\Rightarrow$ Butterworth response
$Q>1/\sqrt{2}\Rightarrow$ Peak occurs</td><td>
</td><td>
</td></tr>
<tr><td>(b) High-pass (HP)
$T(s)=\dfrac{a_2 s^2}{s^2+s(\omega_0/Q)+\omega_0^2}$
$\text{High-frequency gain}=a_2$
$Q>\dfrac{1}{\sqrt{2}}\Rightarrow$ Peak occurs</td><td>
</td><td>
</td></tr>
</table>

Reading Assignment: Second-Order Filter Function (Cont.)

Filter Type and T(s)	S-Plane poles/zeros	|T|
(c) Band-pass (BP) $T(s)=\dfrac{a_1 s}{s^2+s(\omega_0/Q)+\omega_0^2}$ Center-frequency gain $=a_1 Q/\omega_0$ ω_0 : center frequency ω_1,ω_2 : 3-dB value $\omega_1,\omega_2=\omega_0\sqrt{1+(1/4Q^2)}$ $\mp(\omega_0/2Q)$ $BW=\omega_2-\omega_1=\omega_0/Q$		
(d) Notch $T(s)=a_2\dfrac{s^2+\omega_0^2}{s^2+s(\omega_0/Q)+\omega_0^2}$ dc gain $=a_2$ High-frequency gain $=a_2$		

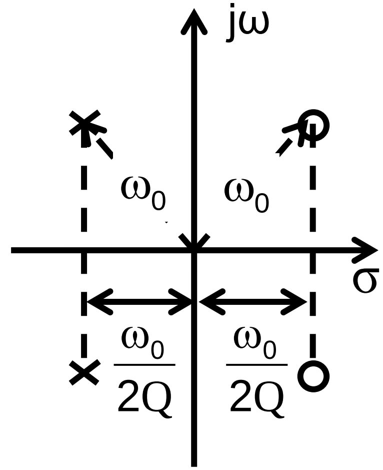
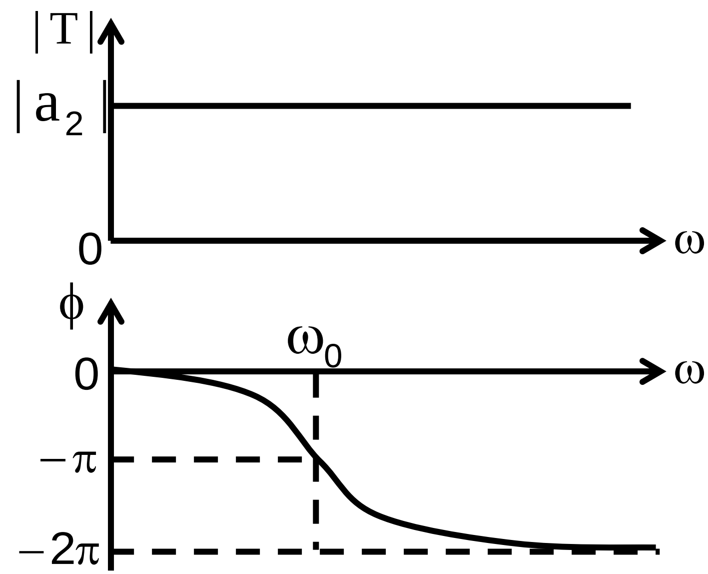
Reading Assignment: Second-Order Filter Function (Cont.)

Filter Type and T(s)	S-Plane poles/zeros	|T|
(e) Low-Pass Notch (LPN) $T(s) = a_2 \frac{s^2 + \omega_n^{\ 2}}{s^2 + s(\omega_0/Q) + \omega_0^2}$ $\omega_n \ge \omega_0$ $\text{dc gain} = a_2 \frac{\omega_n^2}{\omega_0^2}$ $High-\ frequency\ gain = a_2$		 $\omega_{max} = \omega_0 \sqrt{\frac{(\omega_n^2/\omega_0^2)(1-(1/2Q^2))-1}{(\omega_n^2/\omega_0^2)+(1/2Q^2)-1}}$ $T_{max} = \|a_2\| \frac{\|\omega_n^2 - \omega_{max}^2\|}{\sqrt{(\omega_0^2-\omega_{max}^2)^2 + (\omega_0/Q)^2\omega_{max}^2}}$
(f) High-Pass Notch (HPN) $T(s) = a_2 \frac{s^2 + \omega_n^{\ 2}}{s^2 + s(\omega_0/Q) + \omega_0^2}$ $\omega_n \le \omega_0$ $\text{dc gain} = a_2 \frac{\omega_n^2}{\omega_0^2}$ $High-\ frequency\ gain = a_2$		

Reading Assignment: Second-Order Filter Function (Cont.)

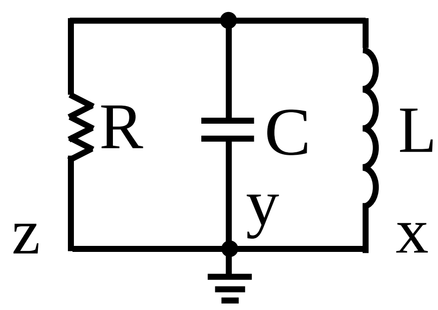
<table>
<tr><th>Filter Type and T(s)</th><th>S-Plane poles/zeros</th><th>|T|</th></tr>
<tr>
<td>(g) All-pass

$T(s) = a_2 \frac{s^2 - s(\omega_0/Q) + \omega_0^2}{s^2 + s(\omega_0/Q) + \omega_0^2}$

Flat gain $= \mathbf{a_2}$</td>
<td>
</td>
<td>
</td>
</tr>
</table>

Second-Order RLC Resonator

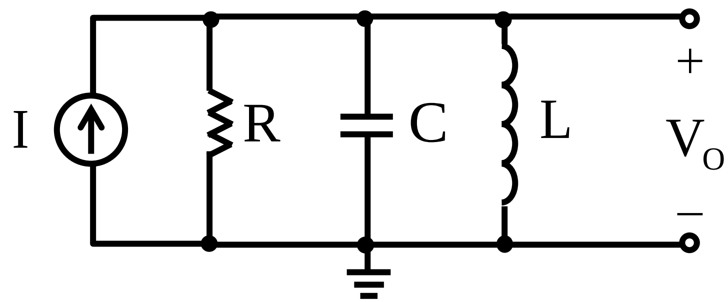
- Use RLC resonator to derive various second-order filter functions.
- Replacing inductor L by a simulated inductor obtained using an OPAMP-RC circuit results in an OPAMP-RC resonator.
- Second-order parallel RLC resonator



  - ◆ Two ways of exciting resonator without changing poles

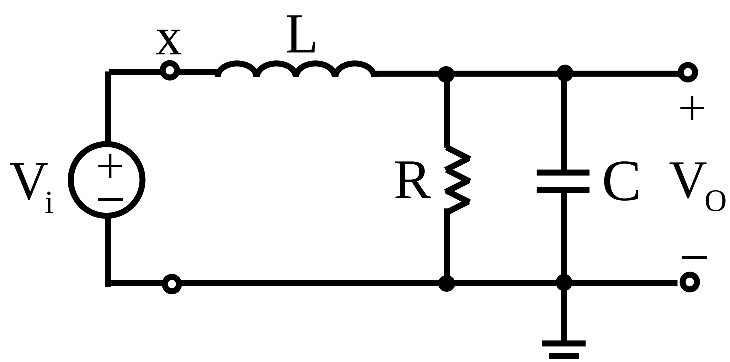
Second-Order RLC Resonator (Cont.)

(a) $\dfrac{V_0}{I} = \dfrac{s/C}{s^2 + s(1/RC) + (1/LC)} = \dfrac{s/C}{s^2 + s(\omega_0/Q) + \omega_0^{\ 2}}$

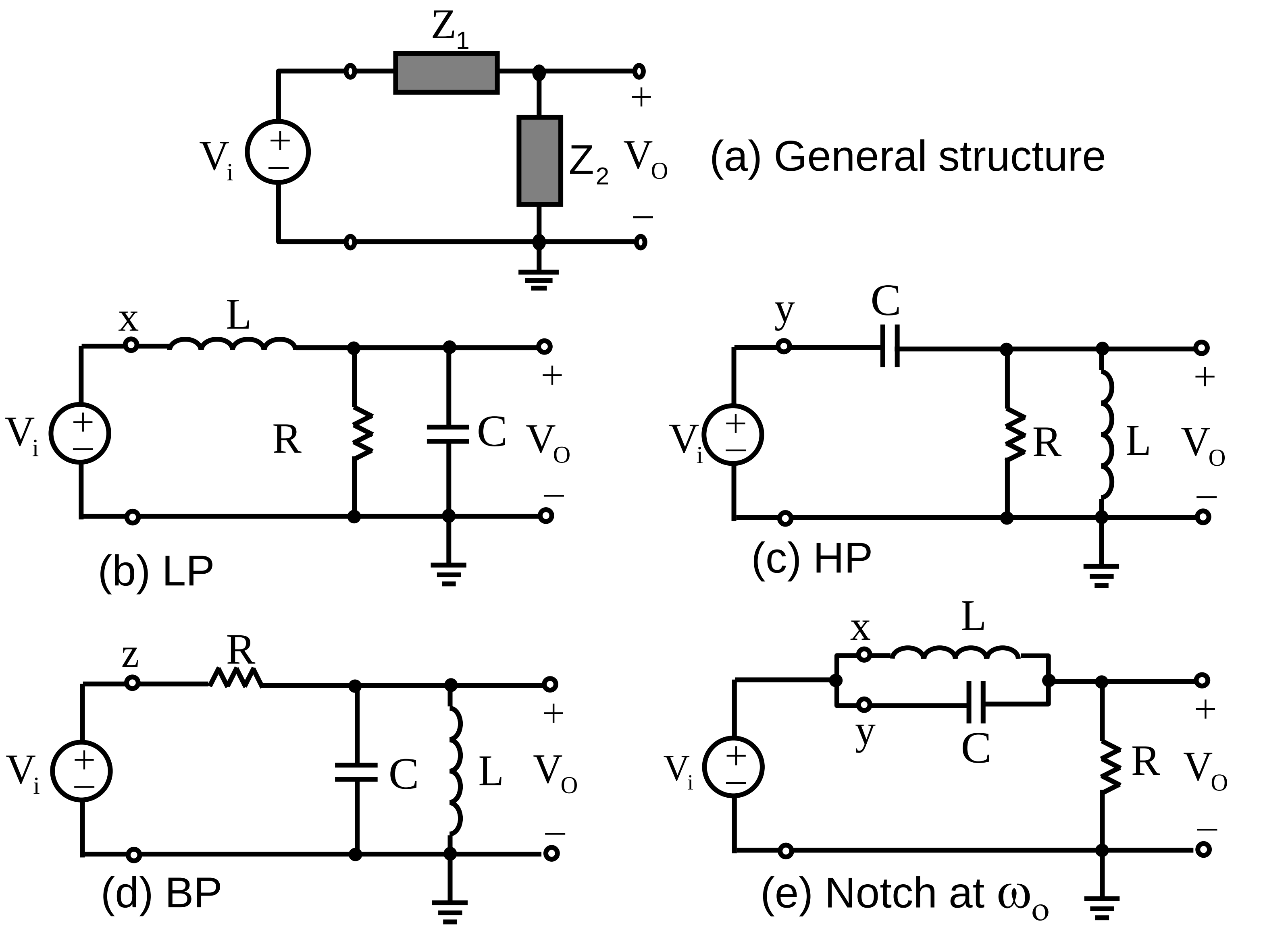


where $\begin{cases} \omega_0 = \dfrac{1}{\sqrt{LC}} \\ Q = \omega_0 RC \end{cases}$

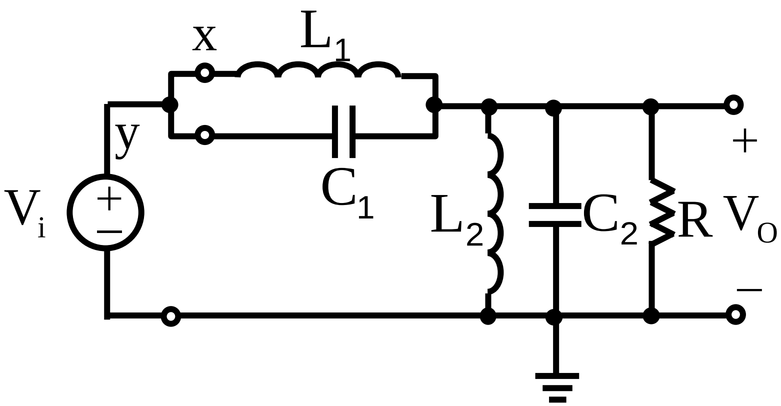
(b) $\dfrac{V_0}{V_i} = \dfrac{1/LC}{s^2 + s(1/RC) + (1/LC)}$



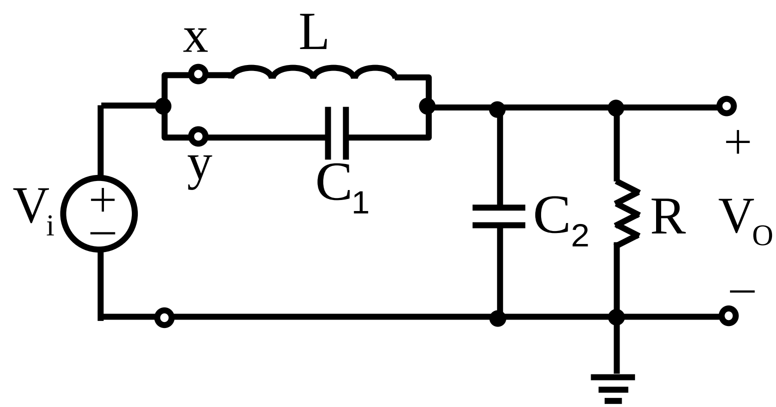
Various Second-Order Functions Using Resonator



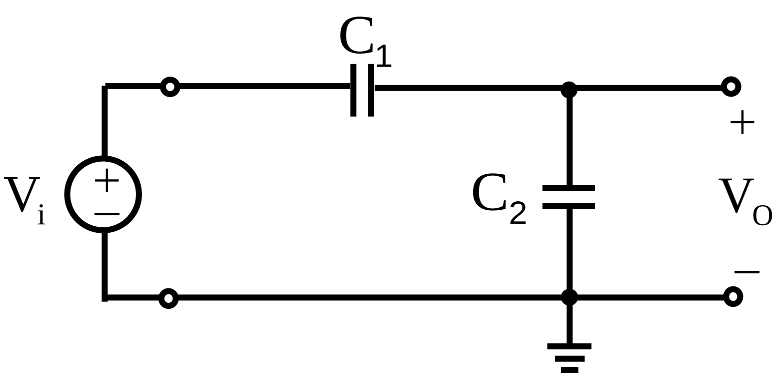
Various Second-Order Functions Using Resonator (Cont.)



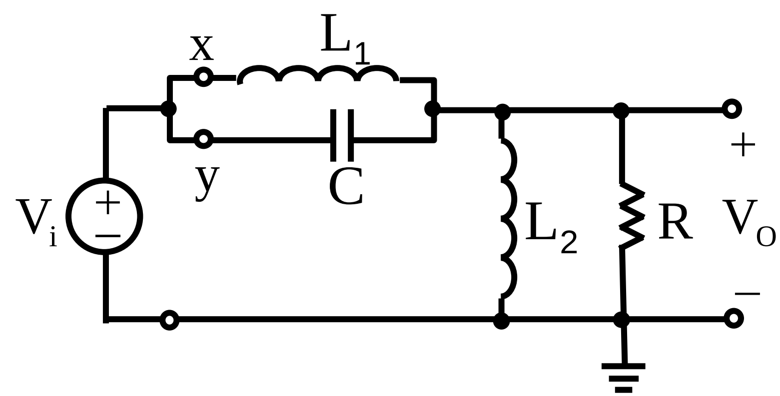
(f) General notch



(g) LPN ($\omega_n > \omega_o$)



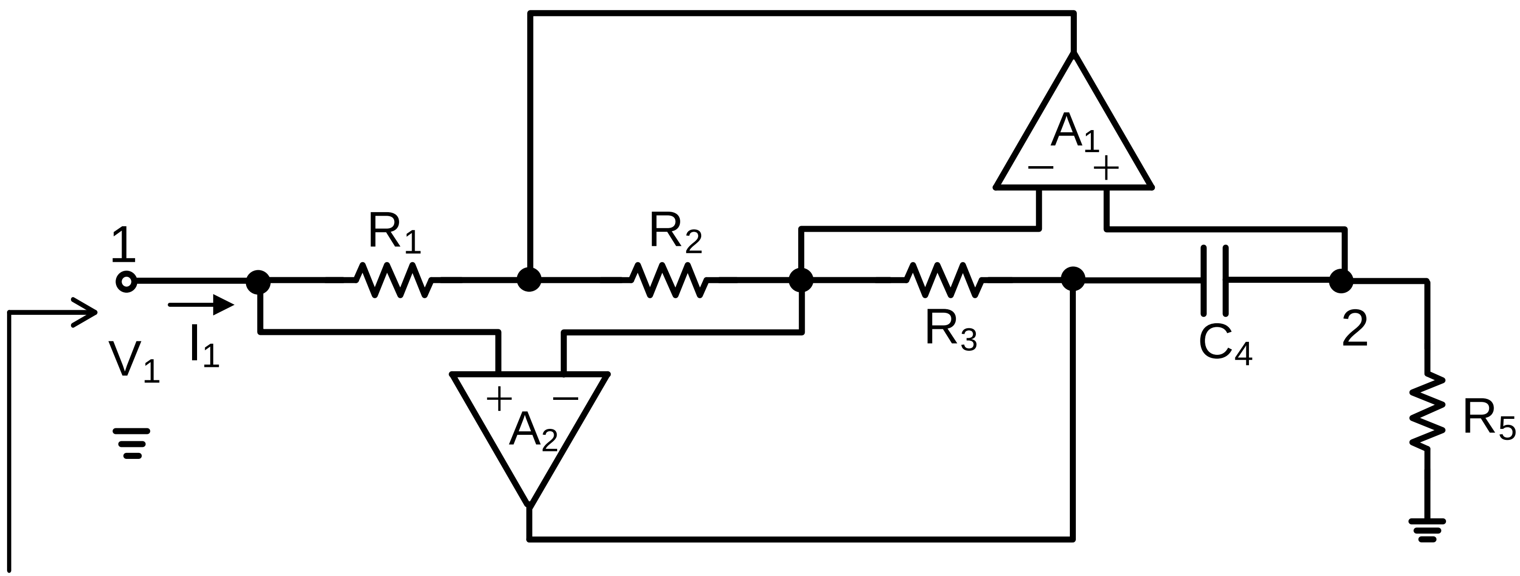
(h) LPN as $s \rightarrow \infty$



(i) HPN ($\omega_n < \omega_0$)

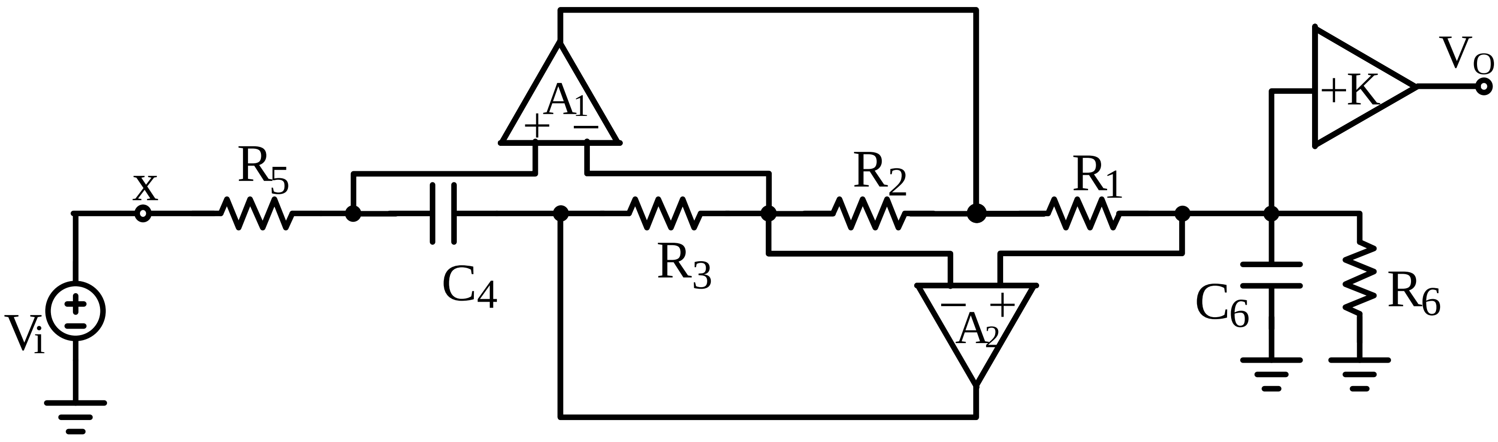
Second-Order Active Filters Based on Inductor Replacement

- Many inductor replacement circuit exists
  - ❖ Antoniou inductance simulation circuit is one of the best, i.e., it is very tolerant to the nonideal properties of the OPAMP gain and bandwidth

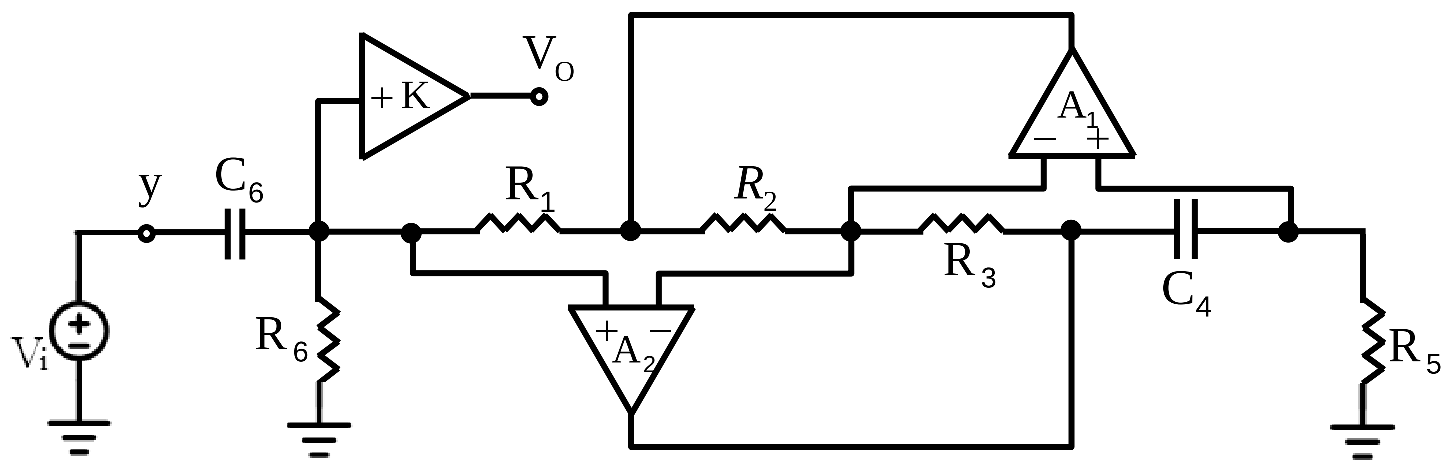


$$Z_{in} \equiv \frac{V_1}{I_1} = sC_4R_1R_3R_5/R_2 \;\Rightarrow L = C_4R_1R_3R_5/R_2$$

OPAMP-RC Resonator Examples



(a)LP

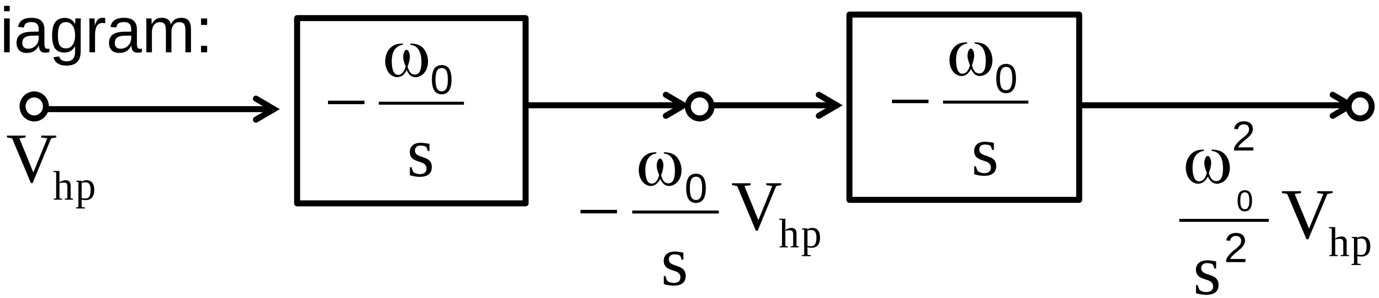


(b)HP

Two-Integrator-Loop Biquad

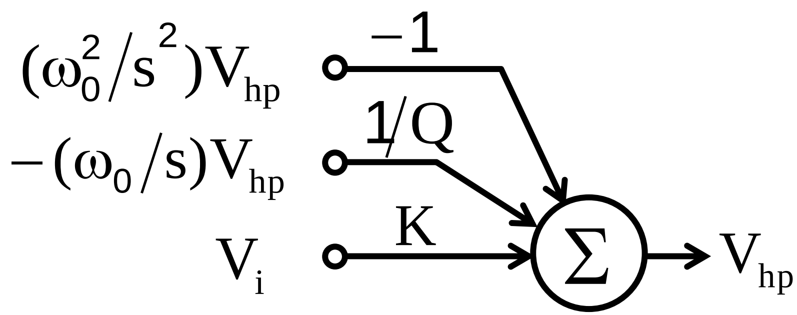
- Derivation
  - High-pass $\Rightarrow \dfrac{V_{hp}}{V_i} = \dfrac{Ks^2}{s^2 + s(\omega_0/Q) + \omega_0^2}$

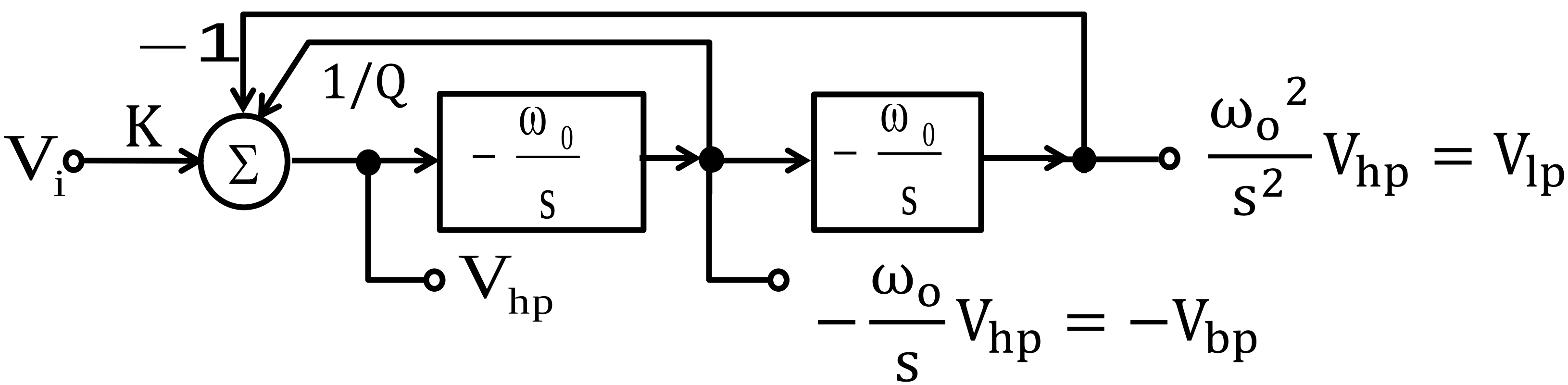
    $\Rightarrow V_{hp} = KV_i - \dfrac{1}{Q}\dfrac{\omega_0}{s}V_{hp} - \dfrac{\omega_0^2}{s^2}V_{hp}$
  - Band-pass $\Rightarrow \dfrac{V_{bp}}{V_i} = \dfrac{K\omega_0 s}{s^2 + s(\omega_0/Q) + \omega_0^2} = \dfrac{\omega_0}{s}\cdot\dfrac{V_{hp}}{V_i}$
  - Low-pass $\Rightarrow \dfrac{V_{lp}}{V_i} = \dfrac{K\omega_0^2}{s^2 + s(\omega_0/Q) + \omega_0^2} = \dfrac{\omega_0^2}{s^2}\cdot\dfrac{V_{hp}}{V_i}$
- Block diagram:



Two-Integrator-Loop Biquad (Cont.)

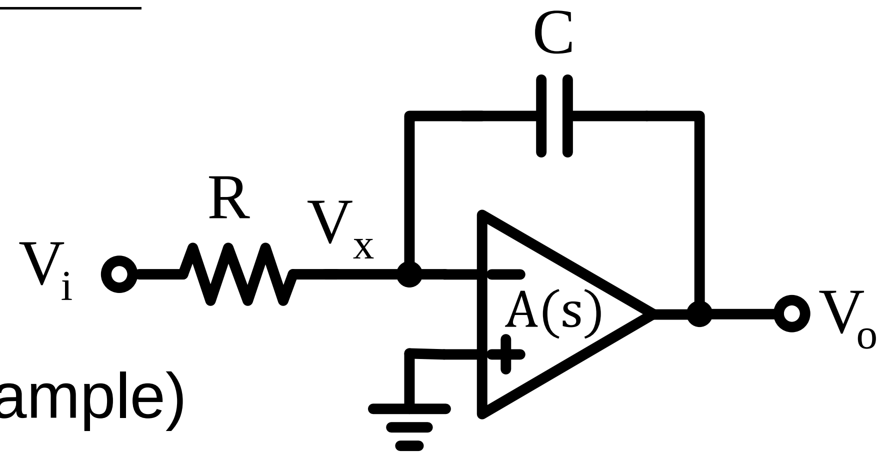
- Universal active filter realizes LP, BP, and HP simultaneously
  - ◆ Versatile
  - ◆ Easy to design
  - ◆ Very popular





❖ Performance limited by the finite BW of OPAMP

Integrator



- Ideal OPAMP (A(s) = ∞)

$$A_V(s) = \frac{V_o(s)}{V_i(s)} = -\frac{1}{sRC}$$

- Actual OPAMP (one-pole example)

$$A(s) = \frac{A_O}{1 + s/s_1}$$ – OPAMP transfer function

If $R_i \to \infty, R_O = 0,$

$$V_x(s) = \frac{-V_o(s)}{A(s)} \Rightarrow \frac{V_i(s) - V_x(s)}{R} + sC\left(V_o(s) - V_x(s)\right) = 0$$

$$A_v(s) = \frac{V_o(s)}{V_i(s)} = \frac{-1}{sRC + \frac{(1 + sRC)}{A(s)}}$$

Integrator (Cont.)

Let $A(s) = \dfrac{A_O}{1 + s/s_1}$,

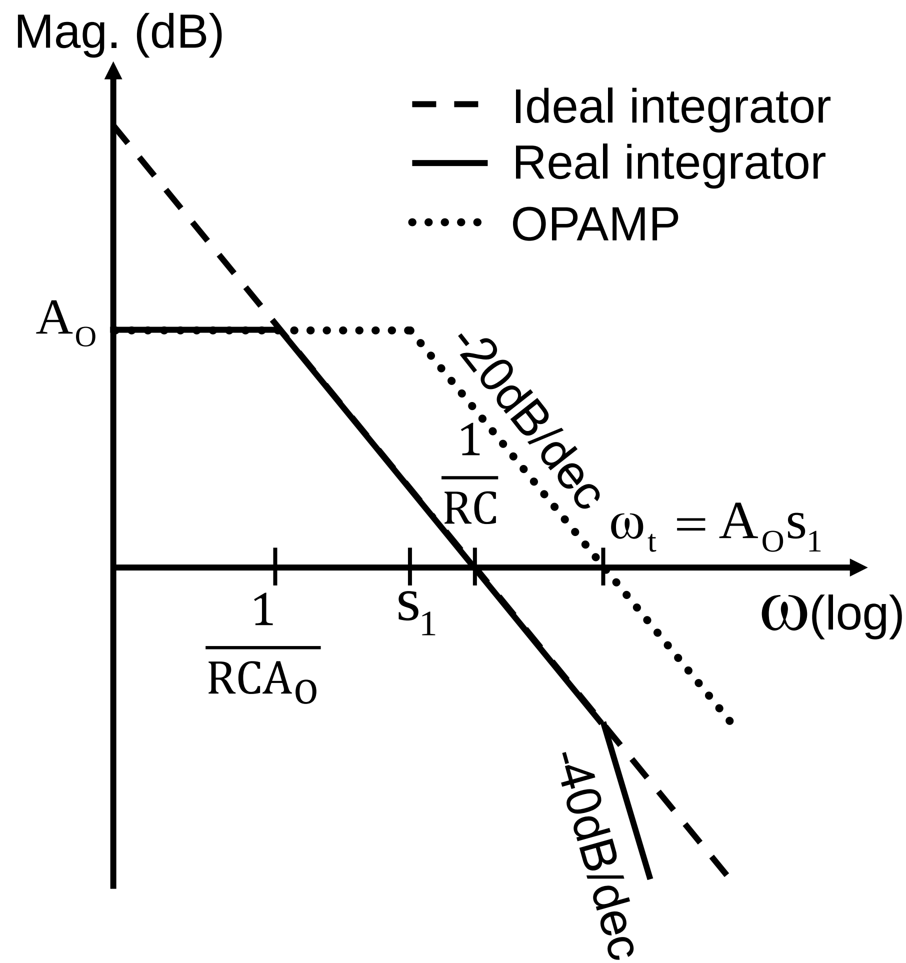
$$A_v(s) = \frac{-1}{\frac{RC}{\omega_t}[s^2 + s(\omega_t + s_1 + \frac{1}{RC}) + \frac{s_1}{RC}]}$$

Since $\omega_t = A_0 s_1 >> \dfrac{1}{RC}, s_1$ and $\dfrac{1}{A_O RC}$

$$\Rightarrow \omega_t + s_1 + \frac{1}{RC} \approx \omega_t \approx \omega_t + \frac{1}{A_O RC}$$

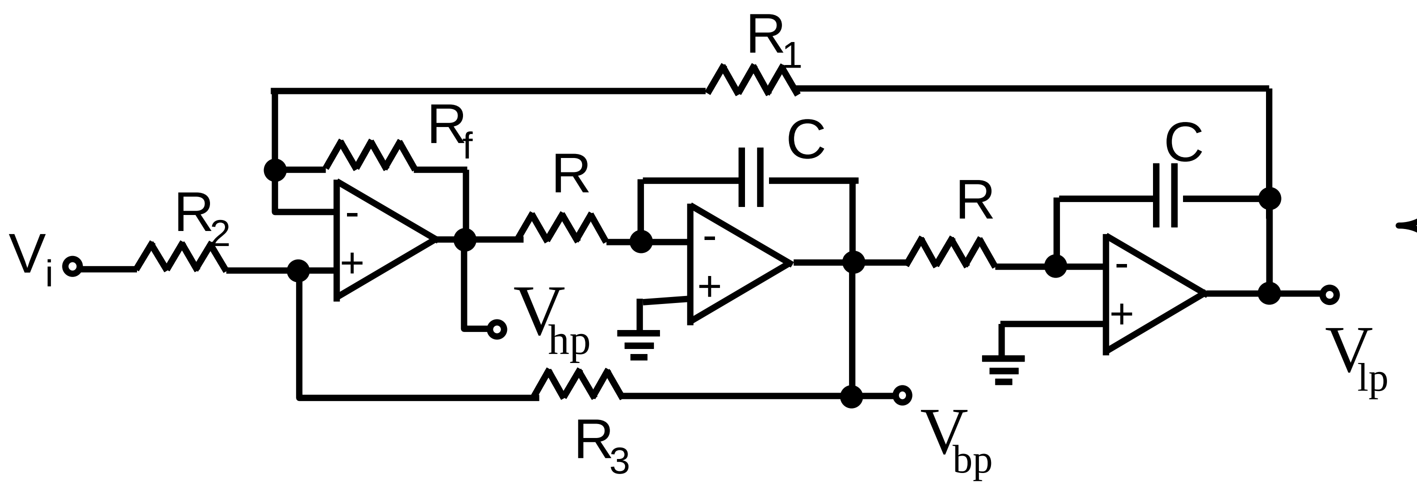
$$\Rightarrow A_v(s) = \frac{-1}{\frac{RC}{\omega_t}(s + \frac{1}{A_O RC})(s + \omega_t)}$$

$$= \frac{-A_O}{(1 + sRCA_O)(1 + \frac{s}{A_O s_1})}$$



Circuit Implementation of Universal Filter

- Kerwin-Huelsman-Newcomb(KHN) biquad

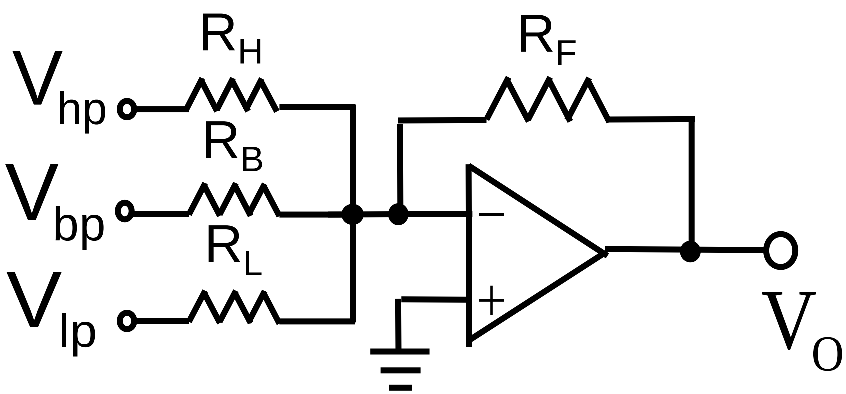


$$\begin{cases} \dfrac{R_f}{R_1} = 1 \\ \dfrac{R_3}{R_2} = 2Q - 1 \end{cases}$$

- ◆ Arbitrarily and practically choose R_1, R_f, R_2, and R_3 to meet the above relationship.

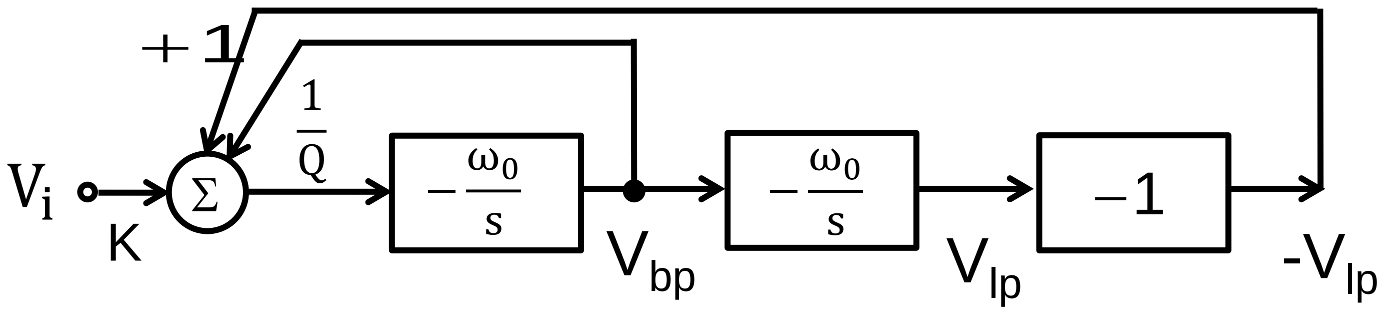
Circuit Implementation of Universal Filter (Cont.)

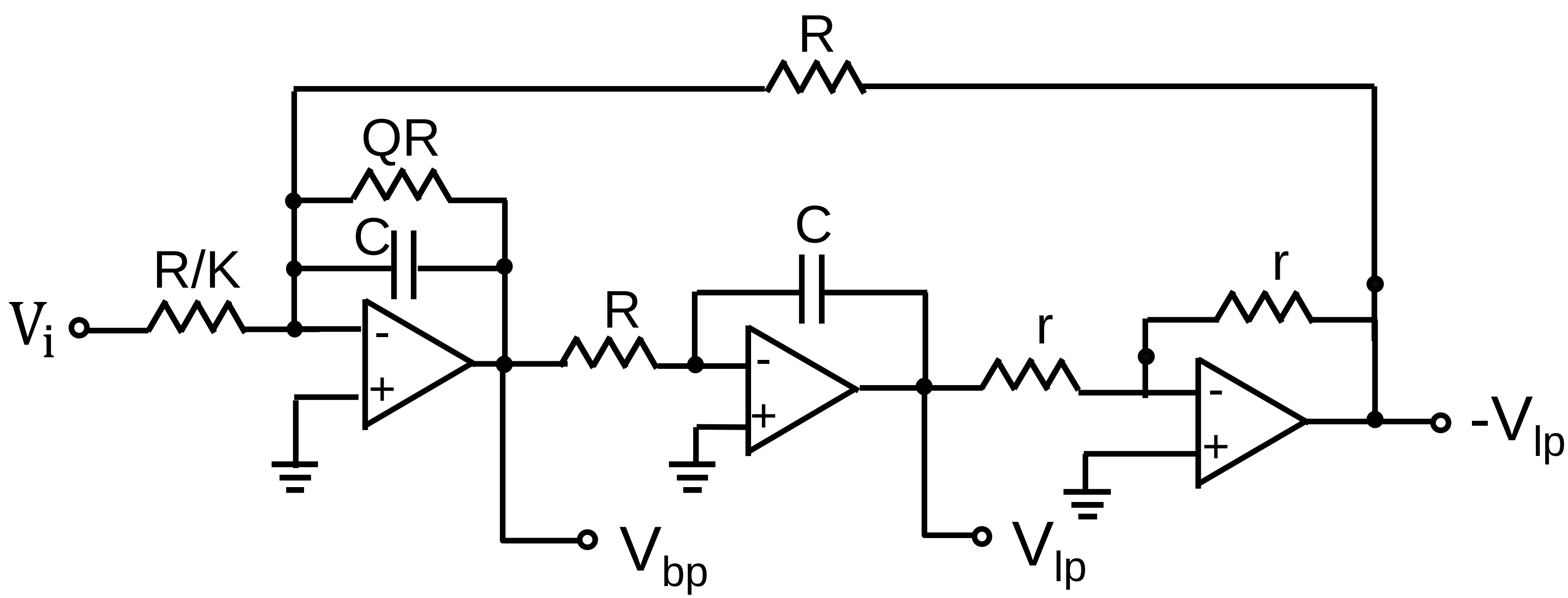
- To obtain notch and all-pass functions, the three outputs are summed with appropriate weight.
  - $V_o = -(\frac{R_F}{R_H}V_{hp} + \frac{R_F}{R_B}V_{bp} + \frac{R_F}{R_L}V_{Lp}) = -V_i(\frac{R_F}{R_H}T_{hp} + \frac{R_F}{R_B}T_{bp} + \frac{R_F}{R_L}T_{Lp})$
  - $\frac{V_0}{V_i} = -K\frac{s^2 - s(R_F/R_B)\omega_0 + (R_F/R_L)\omega_0^{\ 2}}{s^2 + s(\omega_0/Q) + \omega_0^{\ 2}}$



Alternate Two-Integrator-Loop Biquad Circuit

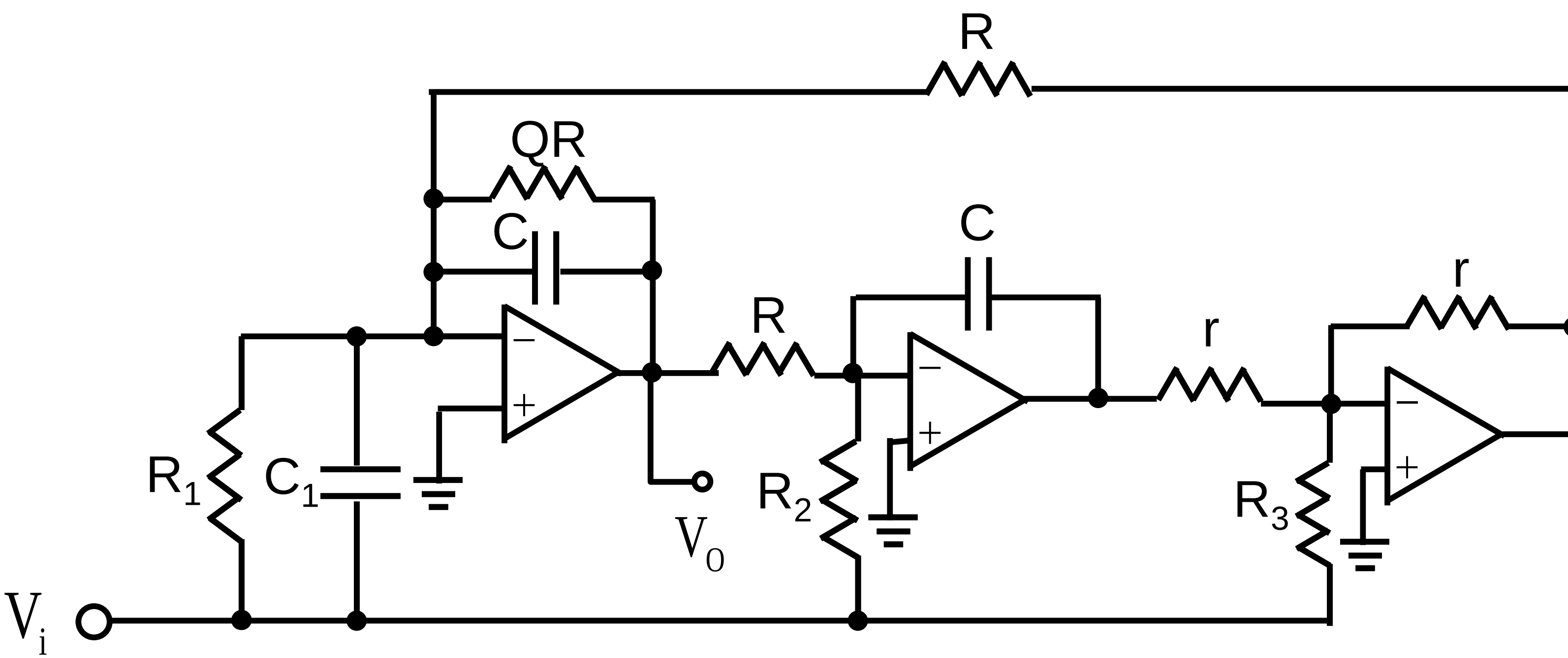
- Tow-Thomas Biquad (Single-ended fashion)
  - Derived from KHN
  - Without V_{hp} output





Alternate Two-Integrator-Loop Biquad Circuit (Cont.)

- Tow-Thomas Biquad (with input feedforward paths)
  - Derived from KHN
  - Can realize all special second-order functions



$$\frac{V_o}{V_i} = -\frac{s^2\left(\frac{C_1}{C}\right) + s\frac{1}{C}\left(\frac{1}{R_1} - \frac{r}{RR_3}\right) + \frac{1}{C^2RR_2}}{s^2 + s\frac{1}{QCR} + \frac{1}{C^2R^2}}$$

Alternate Two-Integrator-Loop Biquad Circuit (Cont.)

- Design Table:

All cases	$C = \text{arbitrary}, R = 1/\omega_0 C, r = \text{arbitrary}$
LP	$C_1 = 0, R_1 = \infty, R_2 = R/\text{dc gain}, R_3 = \infty$
Positive BP	$C_1 = 0, R_1 = \infty, R_2 = \infty, R_3 = Qr/\text{center-frequency gain}$
Negative BP	$C_1 = 0, R_1 = QR/\text{center-frequency gain}, R_2 = \infty, R_3 = \infty$
HP	$C_1 = C \times$ high-frequency gain, $R_1 = \infty, R_2 = \infty, R_3 = \infty$
Notch (all types)	$C_1 = C \times$ high-frequency gain, $R_1 = \infty$, $R_2 = R(\omega_0/\omega_n)^2/\text{high-frequency gain}, R_3 = \infty$
AP	$C_1 = C \times$ flat gain, $R_1 = \infty, R_2 = R/\text{gain}, R_3 = Qr/\text{gain}$

Source: **table 13.2** in Adel S. Sedra and Kenneth C. Smith, "Microelectronic Circuits", 7th edition, 2016.

Single-OPAMP Biquad (SAB)

(Compared with two-integrator biquad)

- Economic

  Requiring 1 OPAMP instead of 3 or 4

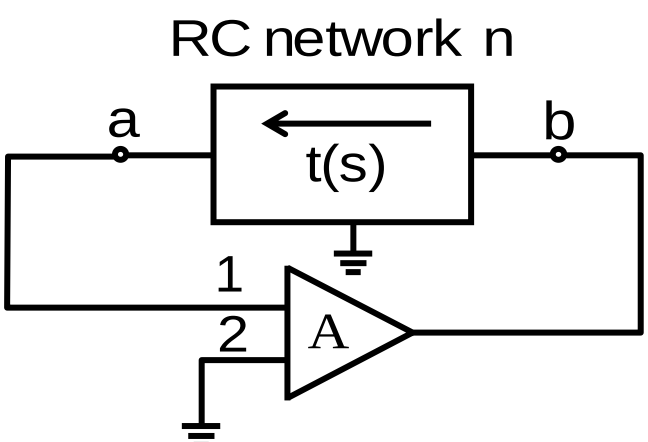
  - ◆ More sensitive to R and C variations
  - ◆ Greater dependence on limited gain and bandwidth

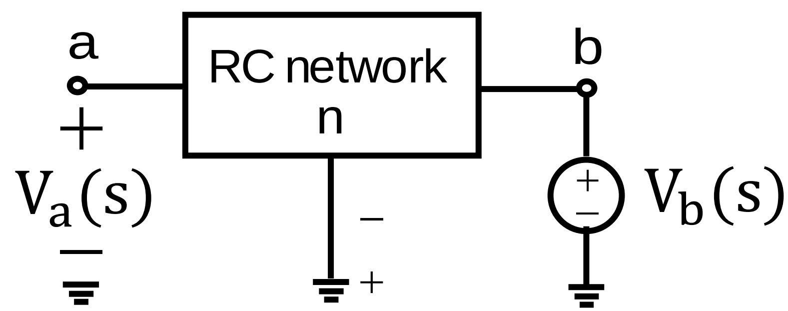
⇨ Single-amplifier biquads (SABs) are therefore limited to the less stringent filter spec, e.g., pole Q<10

- Biquads can be cascaded to construct high-order filters

Reading Assignment: SAB Synthesis

- Two-step procedure
  - Synthesis of a feedback loop that realizes a pair of complex conjugate poles characterized by ω_0 and Q.
  - Injecting the input signal in a way that realizes the desired transmission zeros.
- Synthesis of the feedback loop





$$t(s) \equiv \frac{V_a(s)}{V_b(s)}$$

Reading Assignment: SAB Synthesis (Cont.)

- $t(s) = \dfrac{V_a(s)}{V_b(s)} = \dfrac{N(s)}{D(s)}$
- Loop gain $L(s) = At(s) = \dfrac{AN(s)}{D(s)}$

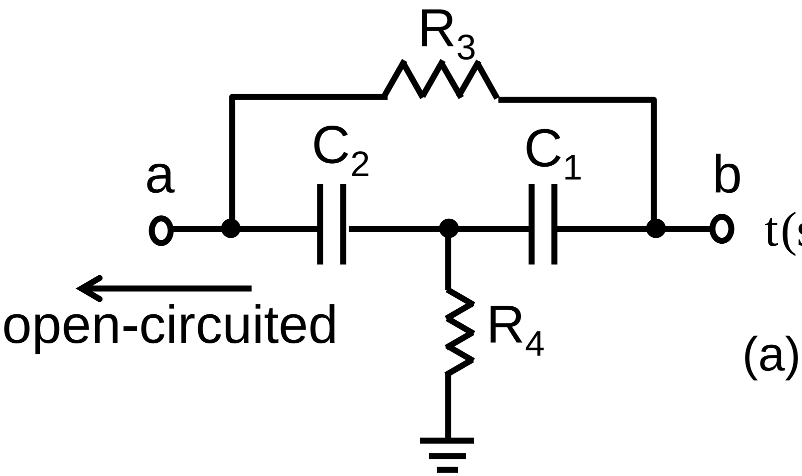
  poles s_p can be obtained from 1+L(s)=0

⇨ $t(s_p) = -\dfrac{1}{A}$

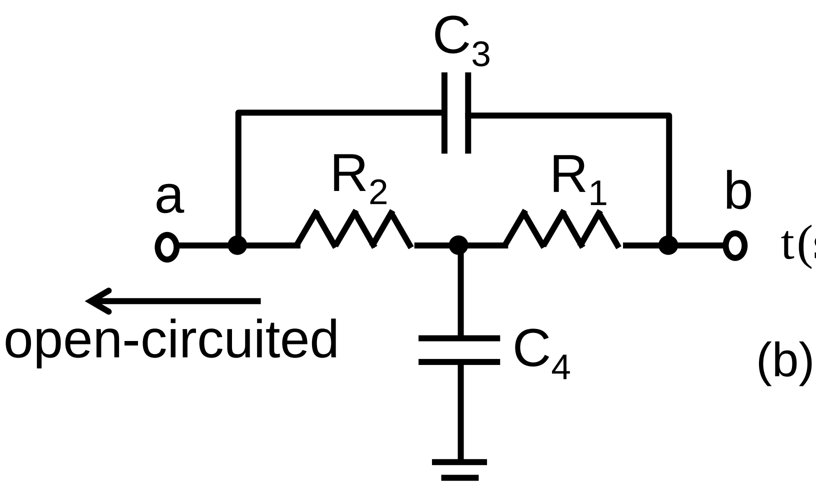
If A>>1, then $N(s_p)=0$

⇨ The circuit poles are identical to the zeros of the RC network

Reading Assignment: Bridged-T RC Networks



$$t(s)=\frac{s^2+s\left(\frac{1}{C_1}+\frac{1}{C_2}\right)\frac{1}{R_3}+\frac{1}{C_1C_2R_3R_4}}{s^2+s\left(\frac{1}{C_1R_3}+\frac{1}{C_2R_3}+\frac{1}{C_1R_4}\right)+\frac{1}{C_1C_2R_3R_4}}$$



$$t(s)=\frac{s^2+s\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{1}{C_4}+\frac{1}{C_3C_4R_1R_2}}{s^2+s\left(\frac{1}{C_4R_1}+\frac{1}{C_4R_2}+\frac{1}{C_3R_2}\right)+\frac{1}{C_3C_4R_1R_2}}$$

Reading Assignment: Bridged-T RC Networks (Cont.)

- The pole polynomial of the active-filter circuit will be equal to the numerator polynomial of the Bridged-T network.

$$s^2 + s\frac{\omega_0}{Q} + \omega_0^2 = s^2 + s\left(\frac{1}{C_1} + \frac{1}{C_2}\right)\frac{1}{R_3} + \frac{1}{C_1 C_2 R_3 R_4}$$

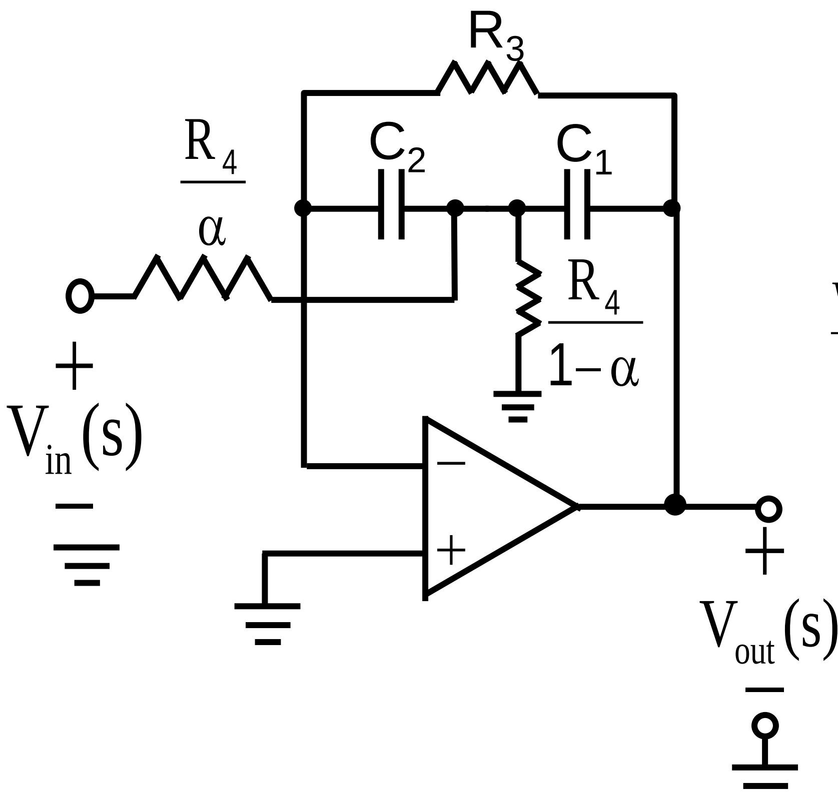
$$\Rightarrow \begin{cases} Q = \left[\dfrac{\sqrt{C_1 C_2 R_3 R_4}}{R_3}\left(\dfrac{1}{C_1} + \dfrac{1}{C_2}\right)\right]^{-1} \\ \omega_0 = \dfrac{1}{\sqrt{C_1 C_2 R_3 R_4}} \end{cases}$$

Let $\begin{cases} C_1 = C_2 = C \\ R_3 = R \\ R_4 = \dfrac{R}{4Q^2} \end{cases}$ $\Rightarrow RC = \dfrac{2Q}{\omega_0}$

$\Rightarrow Q \,\&\, \omega_0$ can be used to determine the component values

Reading Assignment: Bridged-T RC Networks (Cont.)

- Injecting the input signal
  - To obtain transmission zeros
  - Example: a band-pass filter



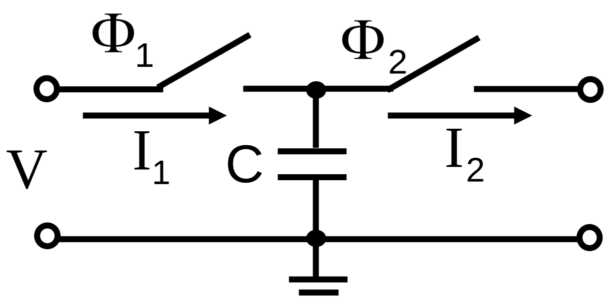
$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{-s\dfrac{\alpha}{C_1R_4}}{s^2 + s\left(\dfrac{1}{C_1} + \dfrac{1}{C_2}\right)\dfrac{1}{R_3} + \dfrac{1}{C_1C_2R_3R_4}}$$

Switched-Capacitor(SC) Filters

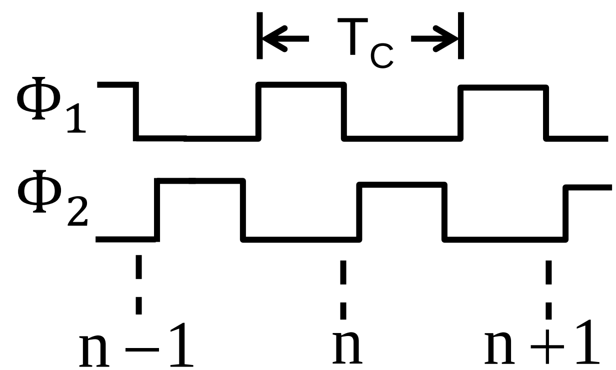
- Switched capacitor performing as a simulated resistor
  - Large area resistor → smaller area capacitor
- VLSI technology requirements
  - Good switch
  - Well-defined capacitor
  - OPAMP
- First discussion on the equivalent resistance of a periodically switched capacitor methods in 1946.
- Rapid evolution of practical SC is due to the implementation in MOS technology.
  - Followed by a rapid development and implementation of analog signal processing techniques
- SC circuits are sampled-data analog MOS integrated circuits

Accurate RC Time Constant

- If $\tau = R_1C_2$, then $\frac{d\tau}{\tau} = \frac{dR_1}{R_1} + \frac{dC_2}{C_2}$
  - The absolute accuracies of R and C is very poor
- C as a resistor
  - Schematic



  - Non-overlapped clock



- For average currents $\overline{I_1}$ and $\overline{I_2}$

$$\overline{I}_1 = \overline{I}_2 = \frac{Q}{T_C} = \frac{CV}{T_C} = \frac{V}{R} \quad \Rightarrow \quad \text{Equivalent } R = \frac{T_C}{C} = \frac{1}{f_C C}$$

Accurate RC Time Constant (Cont.)

- If R_1 is replaced by a switched capacitor resistance realization with a C_1, then

$$\tau = \frac{1}{f_C}\frac{C_2}{C_1} = T_C\frac{C_2}{C_1} \quad \text{and} \quad \frac{d\tau}{\tau} = \frac{dT_C}{T_C} + \frac{dC_2}{C_2} - \frac{dC_1}{C_1}$$

If clock frequency is assumed to be constant,

then $\frac{d\tau}{\tau} = \frac{dC_2}{C_2} - \frac{dC_1}{C_1} \rightarrow 0$

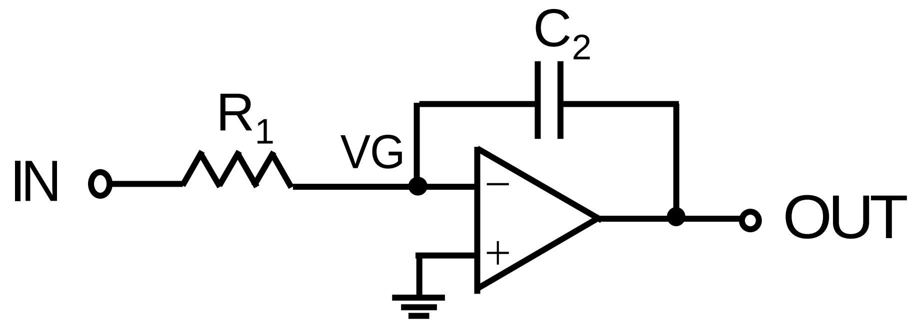
∵ Relative accuracy is good for two capacitors fabricated in the same integrated circuit.

Accurate RC Time Constant (Cont.)

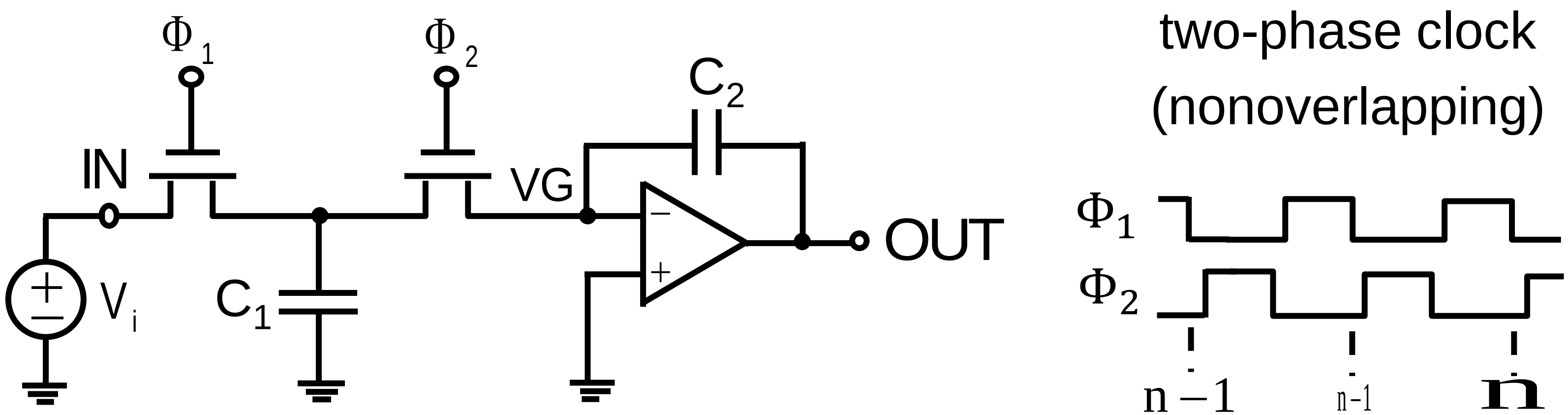
- Basic principle of the switched-capacitor filter technique
  - Active-RC integrator

Transfer function:

$$H(s) = \frac{1}{sR_1C_2}$$



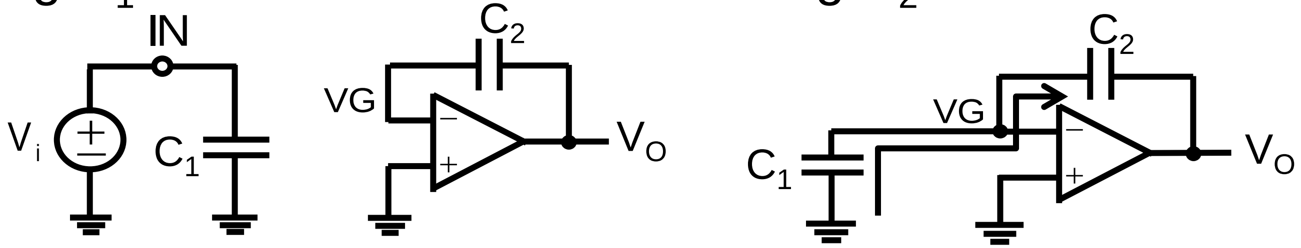
  - Switched-capacitor integrator



Accurate RC Time Constant (Cont.)

- C_1 charges up and then discharges into C_2

During Φ_1 During Φ_2



- Transfer function

$$V_o(n)-V_o(n-1)=-\frac{1}{C_2}\left[Q_{c_2}(n)-Q_{c_2}(n-1)\right]=-\frac{C_1}{C_2}V_i(n-1)$$

After z-transformation, the equation becomes

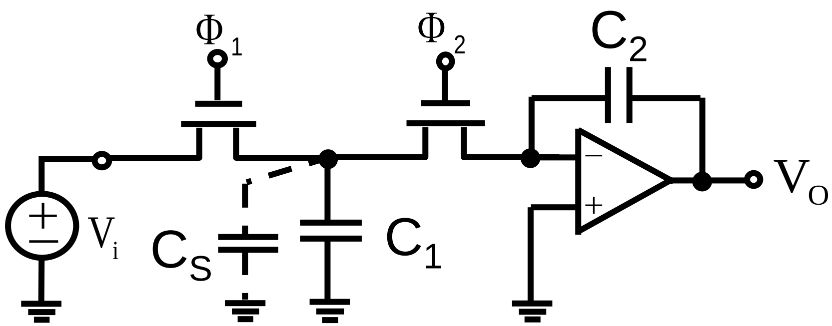
$$\Rightarrow\ V_o(z)-z^{-1}V_o(z)=-\frac{C_1}{C_2}z^{-1}V_i(z)$$

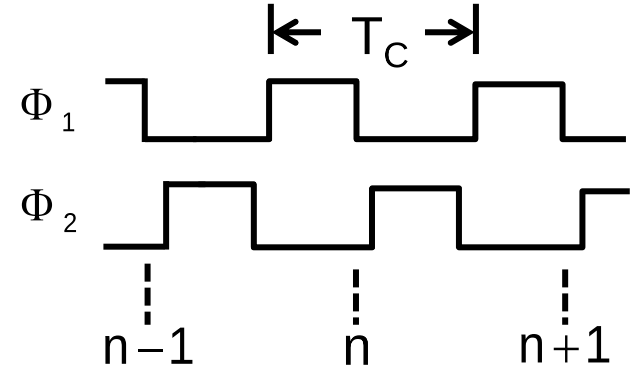
$$H(z)=\frac{V_o(z)}{V_i(z)}=-\frac{C_1}{C_2}\frac{z^{-1}}{1-z^{-1}}$$

simple z-transformation equations
$$\begin{cases} V(n)\rightarrow V(z) \\ V(n-1)\rightarrow z^{-1}V(z) \\ V(n+1)\rightarrow zV(z) \end{cases}$$

Stray-Sensitive SC Integrator

- Stray capacitance





$$R_{eq} = \frac{T_C}{C_1 + C_S}$$; Cs is stray capacitance and voltage dependent

- Transfer function

$$V_O(n) - V_O(n-1) = -\frac{1}{C_2}\left[Q_{C_2}(n) - Q_{C_2}(n-1)\right] = -\frac{1}{C_2}\left[(C_1 + C_S)\,V_i(n-1)\right]$$

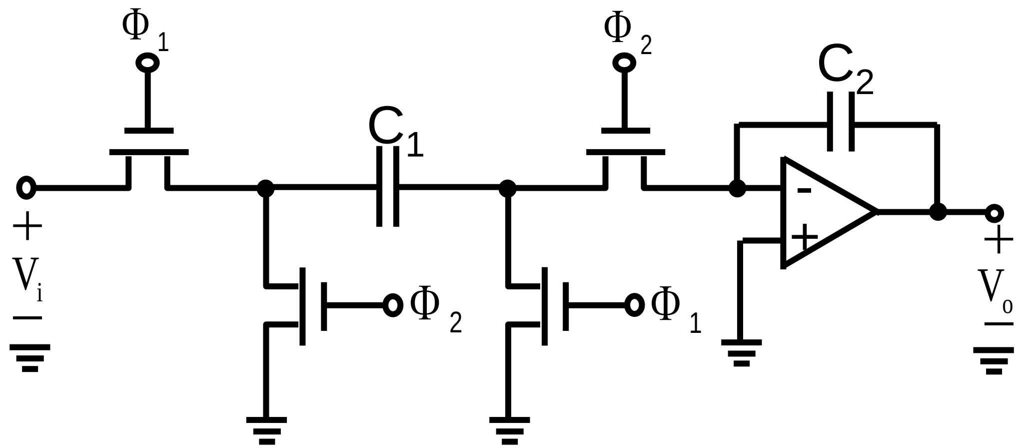
After z-transformation, the equation becomes

$$H(z) = \frac{V_o(z)}{V_i(z)} = -\frac{C_1 + C_S}{C_2}\,\frac{z^{-1}}{1 - z^{-1}}$$

Stray-Insensitive SC Integrators

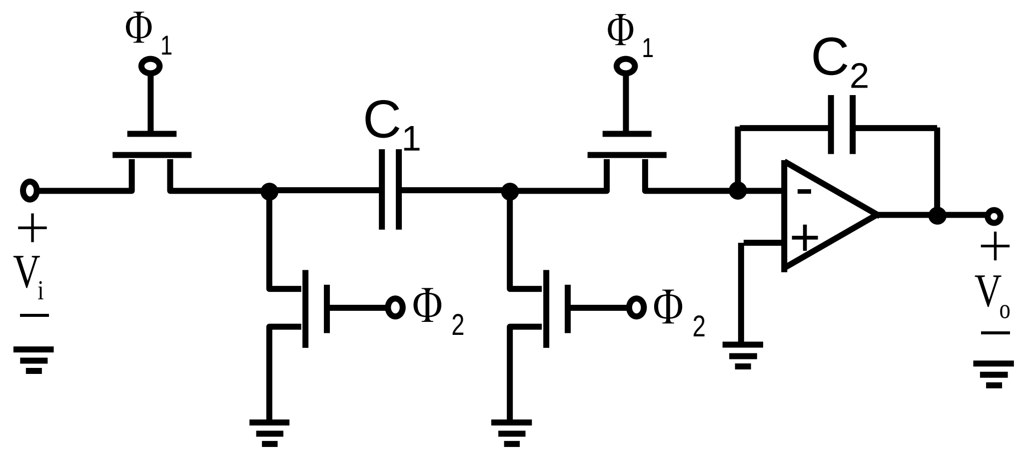
- Noninverting

$$H(z) = \frac{V_o(z)}{V_i(z)} = \frac{C_1}{C_2}\frac{z^{-1}}{1 - z^{-1}}$$



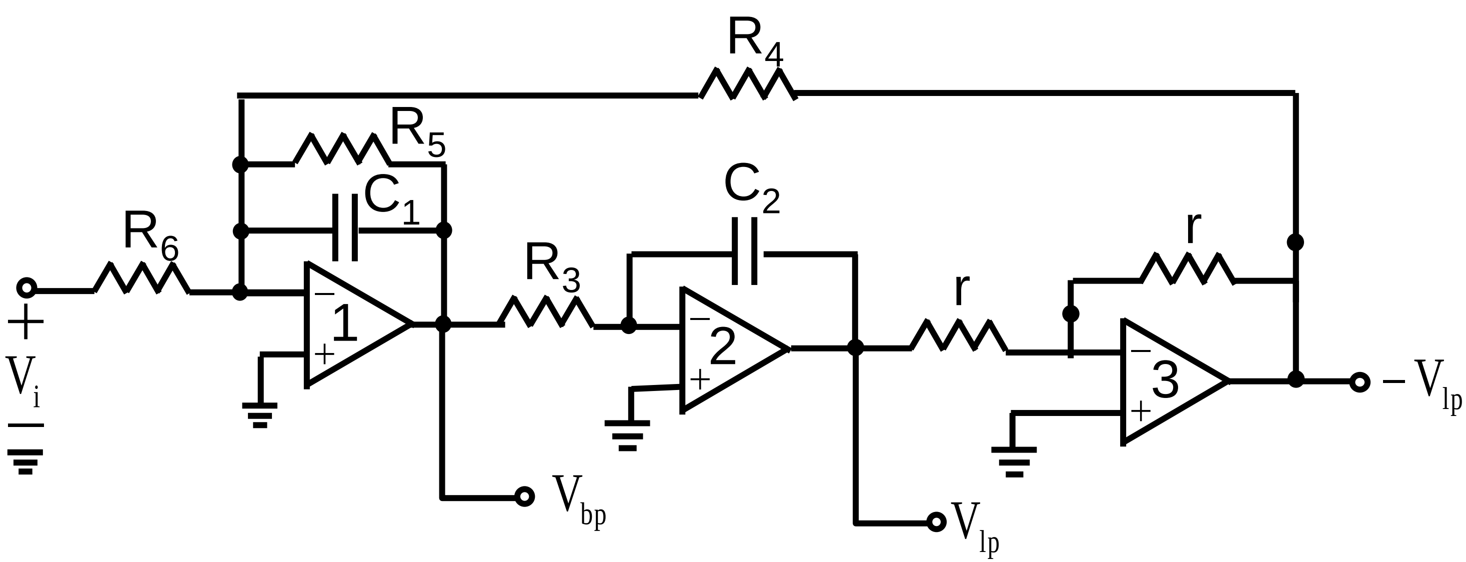
- Inverting

$$H(z) = \frac{V_o(z)}{V_i(z)} = -\frac{C_1}{C_2}\frac{1}{1 - z^{-1}}$$



Reading Assignment: An Active-RC Biquad and Its Switched-Capacitor Counterpart

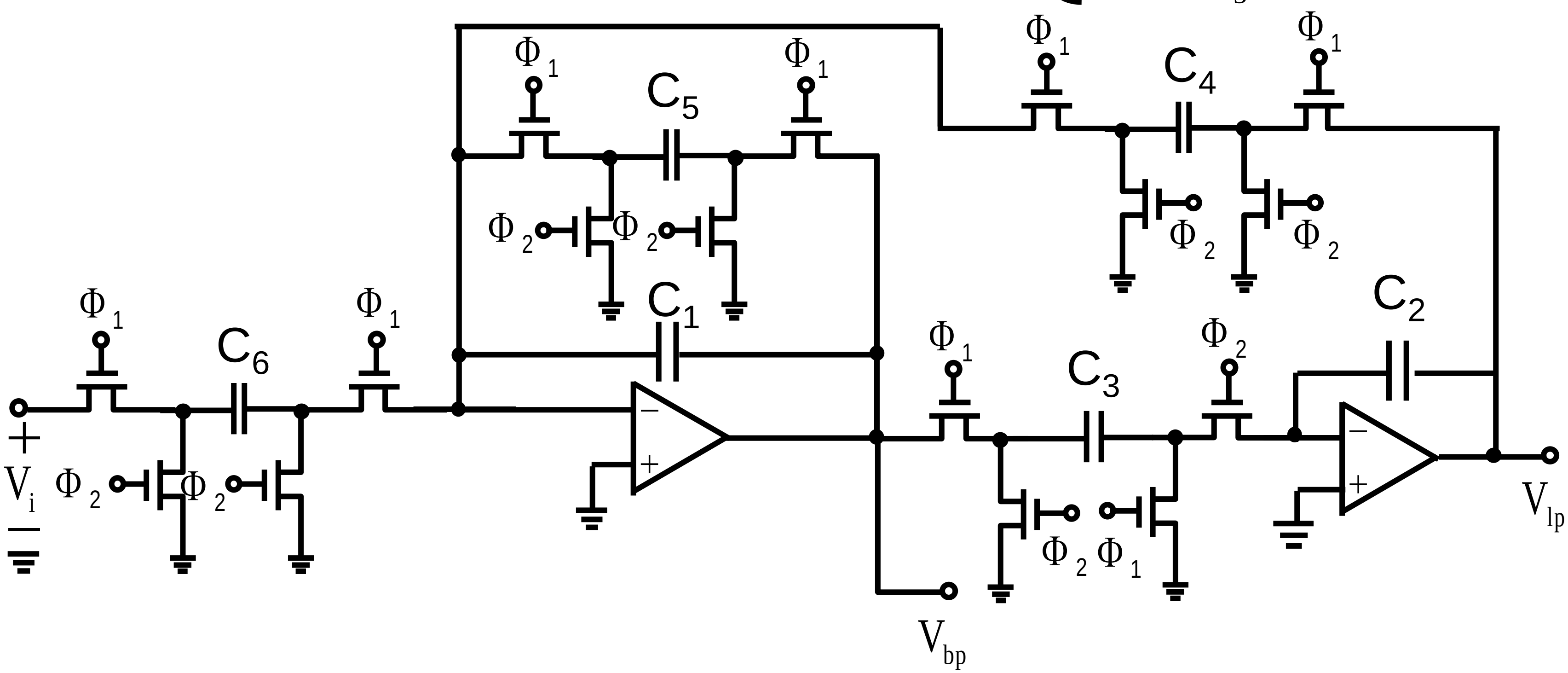
Active RC
$$\begin{cases} \omega_0 = \dfrac{1}{\sqrt{C_1 C_2 R_3 R_4}} \\ Q = \dfrac{R_5}{R_4} \end{cases}$$



Reading Assignment: An Active-RC Biquad and Its Switched-Capacitor Counterpart (Cont.)

- $R_3 = \dfrac{T_C}{C_3}, R_4 = \dfrac{T_C}{C_4}, R_5 = \dfrac{T_C}{C_5}$

  $\Rightarrow$ SC determined by capacitor ratio $\begin{cases} \omega_0 = \dfrac{1}{T_C}\sqrt{\dfrac{C_3 C_4}{C_2 C_1}} \\ Q = \dfrac{C_4}{C_5} \end{cases}$



Filters

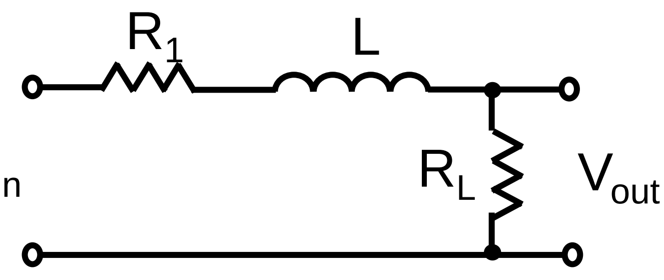
- Continuous-time filter
  - RLC passive
  - RC active

- Sampled-Data filter
  - Switched-Capacitor Filter

- Digital Filter

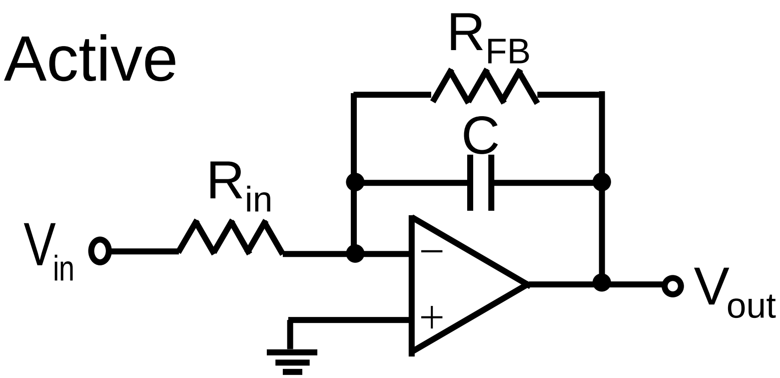
Continuous-Time Filters

- Example: 1 pole low pass filter
  - Passive



$$\frac{V_{out}}{V_{in}} = \frac{R_L}{R_1 + R_L}\left[\frac{1}{1+\frac{sL}{R_1+R_L}}\right]$$

  - Active



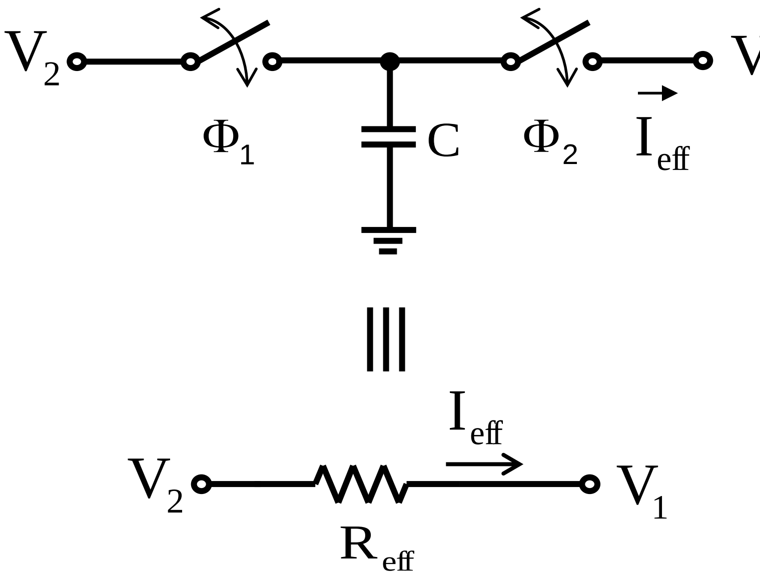
$$\frac{V_{out}}{V_{in}} = -\frac{R_{FB}}{R_{in}}\left[\frac{1}{1+sR_{FB}C}\right]$$

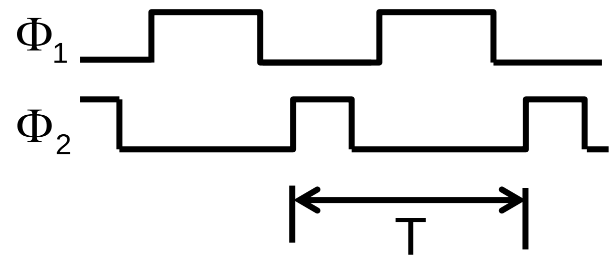
  - Equivalence conditions of the above

$$R_{in} = R_1, R_{FB} = \frac{R_1R_L}{R_1+R_L}, C = \frac{L}{R_1R_L}$$

Switched-Capacitor Filter (SCF)

- Basic concept





$$\Delta Q = C(V_2 - V_1) \quad \Rightarrow \quad I_{eff} = \frac{\Delta Q}{T} = \frac{V_2 - V_1}{R_{eff}} \quad \Rightarrow \quad R_{eff} = \frac{T}{C}$$

SCF (Cont.)

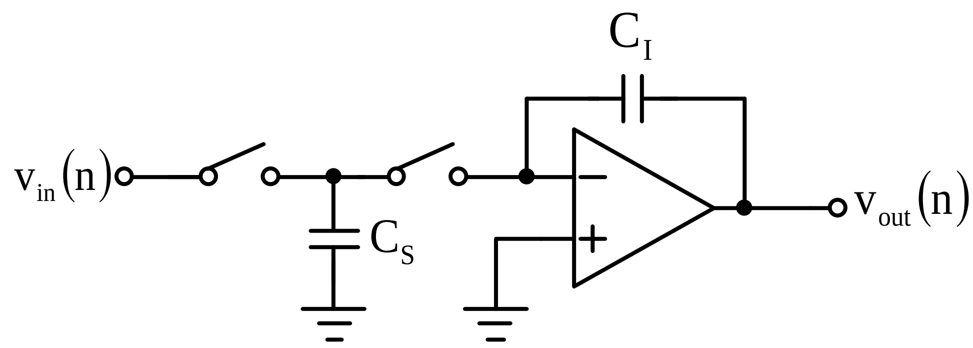
- Example: SC integrator (stray-sensitive)

$$v_{out}(n)-v_{out}(n-1)=-\frac{C_S}{C_I}v_{in}(n-1)$$

$$\Rightarrow\ v_{out}(n)=v_{out}(n-1)-\frac{C_S}{C_I}v_{in}(n-1)$$

$$\Rightarrow\ V_{out}(z)=z^{-1}V_{out}(z)-\frac{C_S}{C_I}z^{-1}V_{in}(z)$$

$$\Rightarrow\ H(z)=\frac{V_{out}(z)}{V_{in}(z)}=-\frac{C_S}{C_I}\frac{z^{-1}}{1-z^{-1}}\quad \text{where}\ \ z=e^{j\omega T}$$



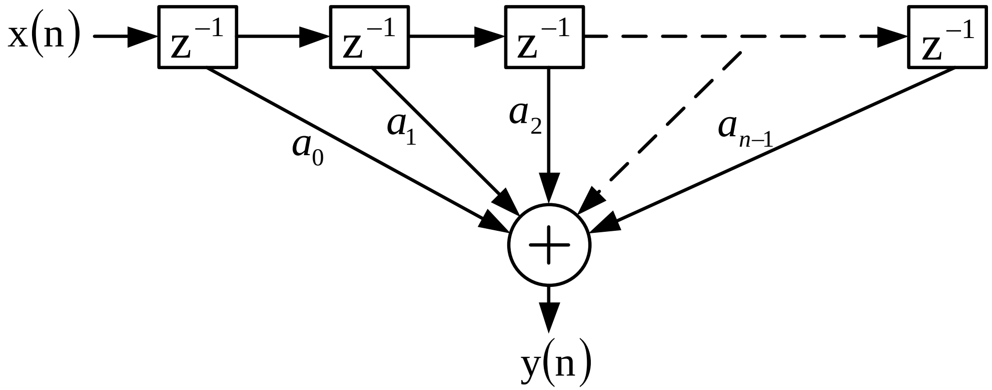
If $\omega T \ll 1$

$$\lim_{\omega T\ll 1}H\left[e^{j\omega T}\right]=-\frac{C_S}{C_I}\frac{e^{-j\omega T}}{1-e^{-j\omega T}}=-\frac{C_S}{C_I}\frac{\left[1-j\omega T+\frac{(j\omega T)^2}{2}-\cdots\cdots\right]}{j\omega T-\frac{(j\omega T)^2}{2}+\cdots\cdots}$$

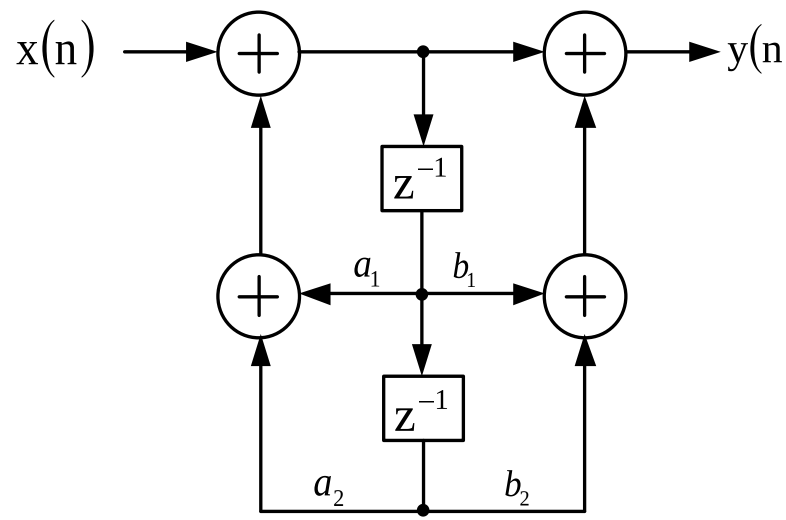
$$\approx-\frac{C_S}{C_I}\frac{1}{j\omega T}=-\frac{1}{j\omega\left(\frac{T}{C_S}\right)C_I}=-\frac{1}{j\omega R_{eff}C_I}$$

Digital Filter

- FIR (Finite Impulse Response)



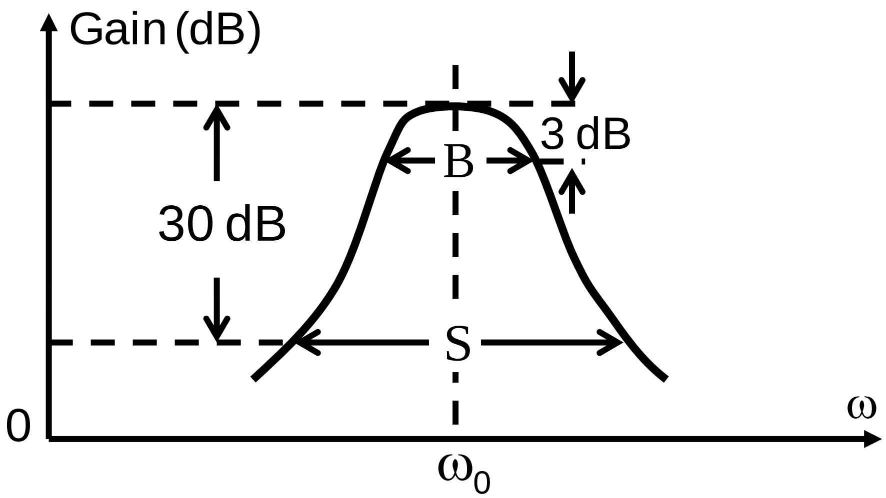
- IIR (Infinite Impulse Response)



- Operations
  - Multiply
  - Delay
  - Add

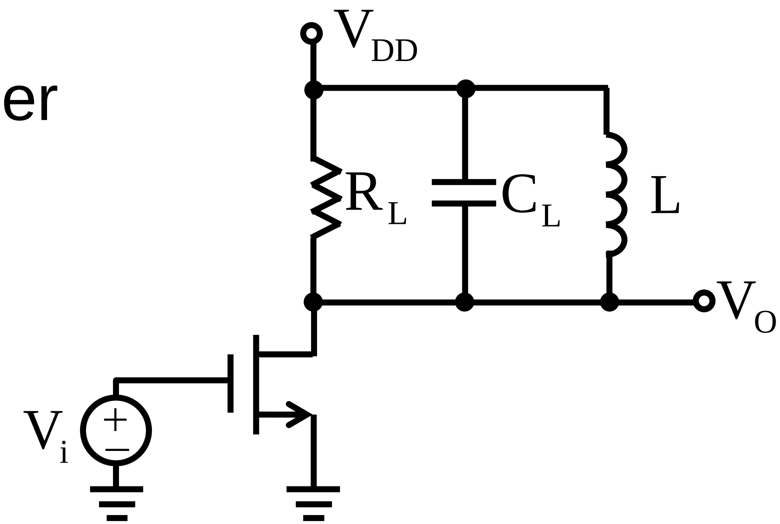
Tuned Amplifiers

- A special kind of frequency-selective network
- RF and IF application
- Response is similar to that of band-pass filters
- Center frequency ranges from hundred kHz to hundred GHz
- Skirt selectivity $\frac{S}{B} = \frac{30dB\ bandwidth}{3dB\ bandwidth}$
  - ◆ Frequency response of a tuned amplifier

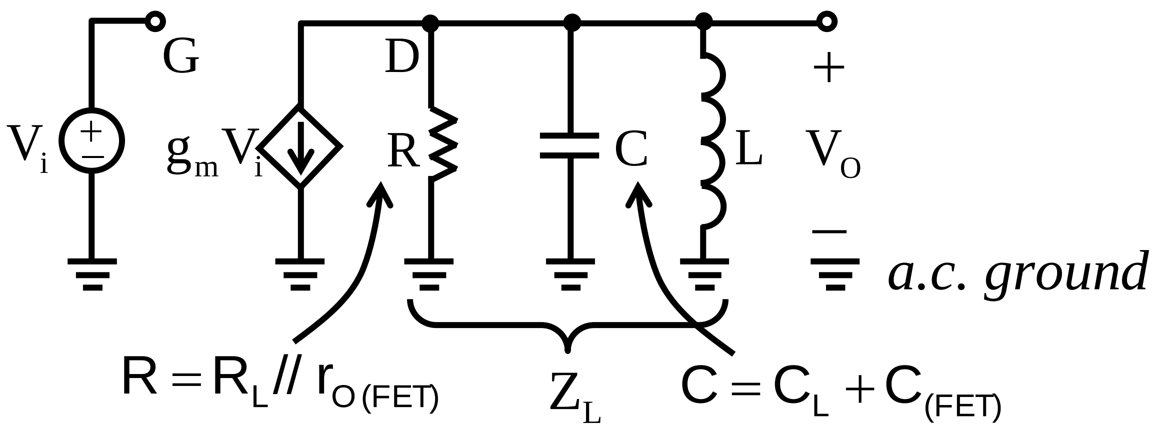


Tuned Amplifiers (Cont.)

- Basic principle
  - Single-tuned amplifier



  - Equivalent circuit



Tuned Amplifiers (Cont.)

$$V_O(s) = -g_m V_i\left(\frac{1}{sC} \parallel R \parallel sL\right) = \frac{-g_m V_i}{sC + \frac{1}{R} + \frac{1}{sL}}$$

$$\frac{V_O(s)}{V_i(s)} = -\frac{g_m}{C}\frac{s}{s^2 + s\left(\frac{1}{RC}\right) + \frac{1}{LC}} \Longrightarrow \begin{cases} \omega_0 = \frac{1}{\sqrt{LC}} \\ B = \frac{1}{RC} \\ Q = \frac{\omega_0}{B} = \omega_0 RC \end{cases}$$

Center frequency gain

$$\frac{V_O(j\omega_0)}{V_i(j\omega_0)} = -\frac{g_m}{C}\frac{j\omega_0}{-\omega_0^2 + j\omega_0(1/RC) + \omega_0^2} = -g_m R$$

Inductor Losses

- Inductor equivalent circuits

Fig.(a)

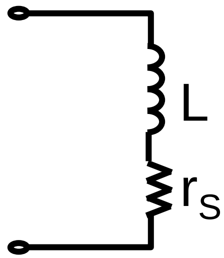
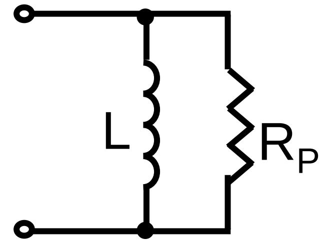


Fig.(b)



- Inductor Q quality factor at the frequency of interest is defined as $Q_0 = \frac{\omega_0 L}{r_s}$; typically $50 \sim 200 \Rightarrow$ Fig.(a)
- Relationship between Q and R_p

$$Y(j\omega_o) = \frac{1}{r_s + j\omega_o L} = \frac{1}{j\omega_o L}\frac{1}{1 - j(\frac{1}{Q_o})} = \frac{1}{j\omega_o L}\frac{1 + j(\frac{1}{Q_o})}{1 + (\frac{1}{Q_o^{\ 2}})}$$

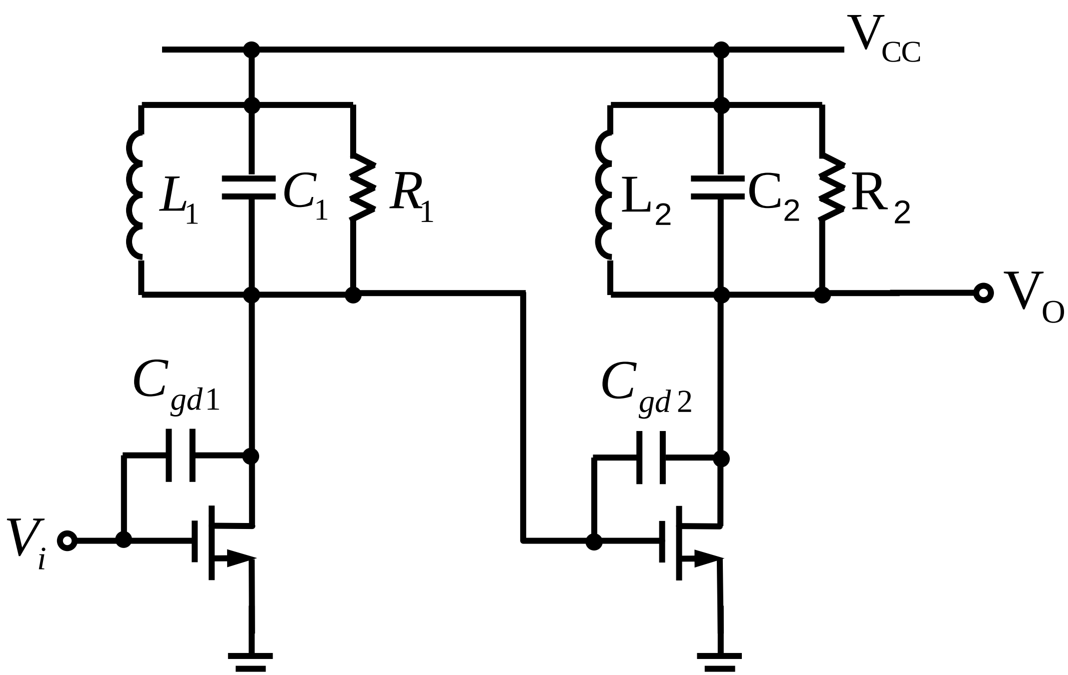
$$\text{For } Q_0 >> 1,\ Y(j\omega_o) \approx \frac{1}{j\omega_o L}(1 + j\frac{1}{Q_o}) = \frac{1}{j\omega_o L} + \frac{1}{\omega_o L Q_o} = \frac{1}{j\omega_o L} + \frac{1}{R_p}$$

$$\Rightarrow R_p = \omega_o L Q_o = \frac{(\omega_o L)^2}{r_s}$$

$\Downarrow$ Fig. (b)

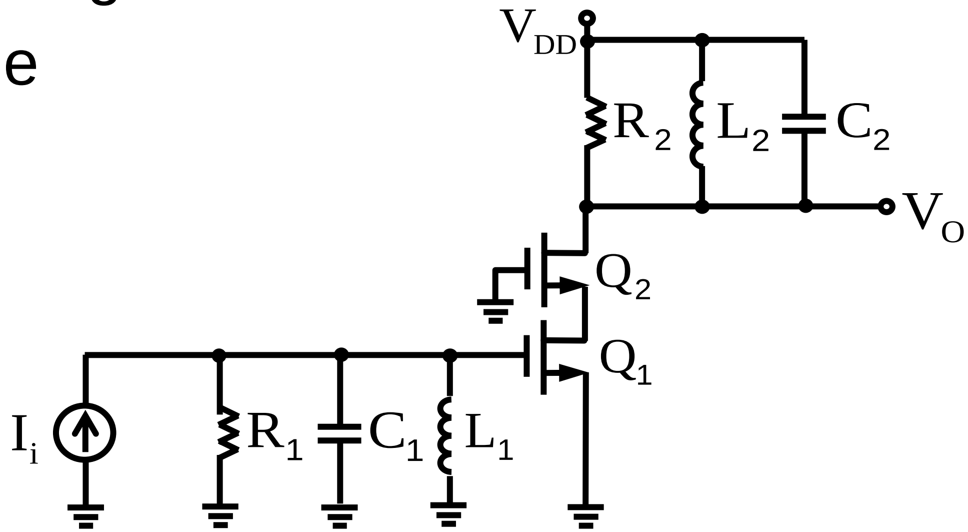
Amplifiers with Multiple Tuned Circuits

- Greater selectivity can be obtained
- Double-tuned amplifier
  - Miller capacitance will cause detuning of the input circuit.

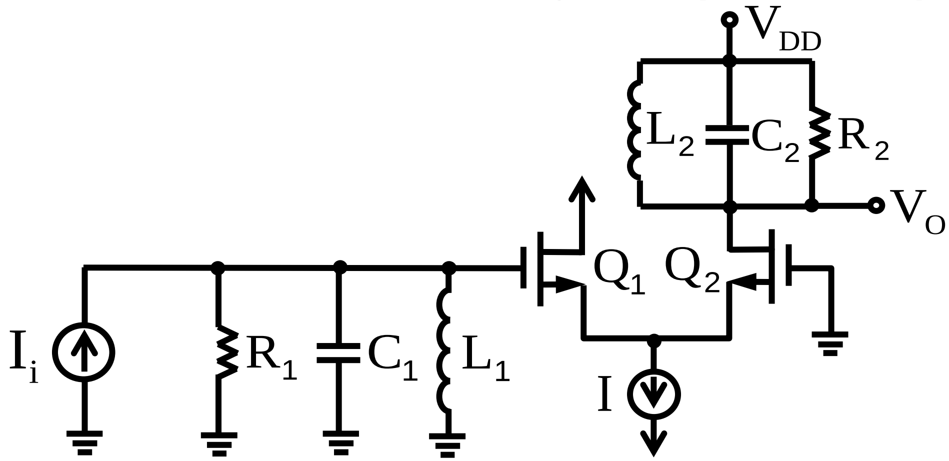


Cascode and CD-CG Cascade

- Amplifier configurations do not suffer from the Miller effect.
  - Cascode



  - Common-drain common-gate (CD-CG) cascade



Synchronous Tuning

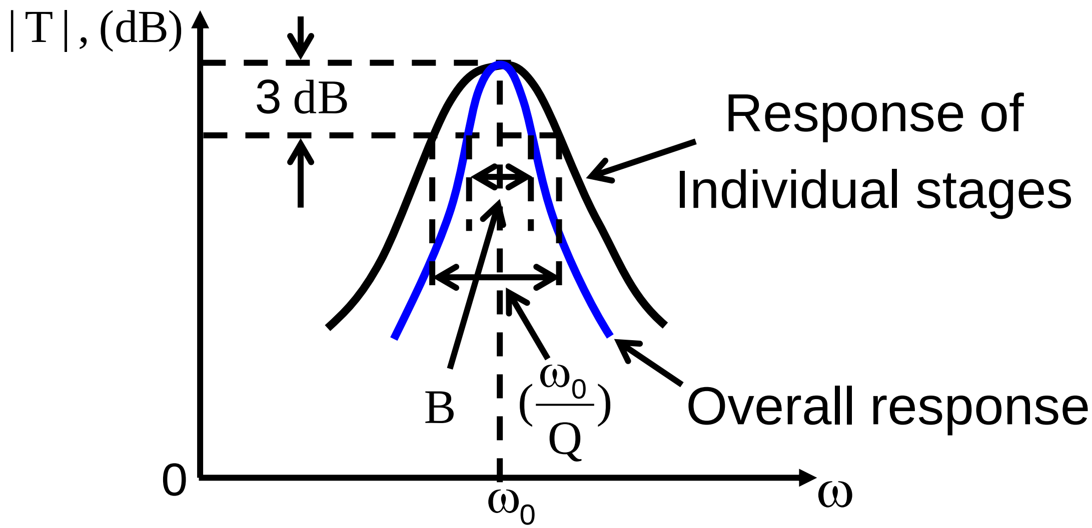
- N identical tuned circuits

  (do not interact)

- Bandwidth shrinkage

$$B = \frac{\omega_0}{Q}\sqrt{2^{1/N} - 1}$$

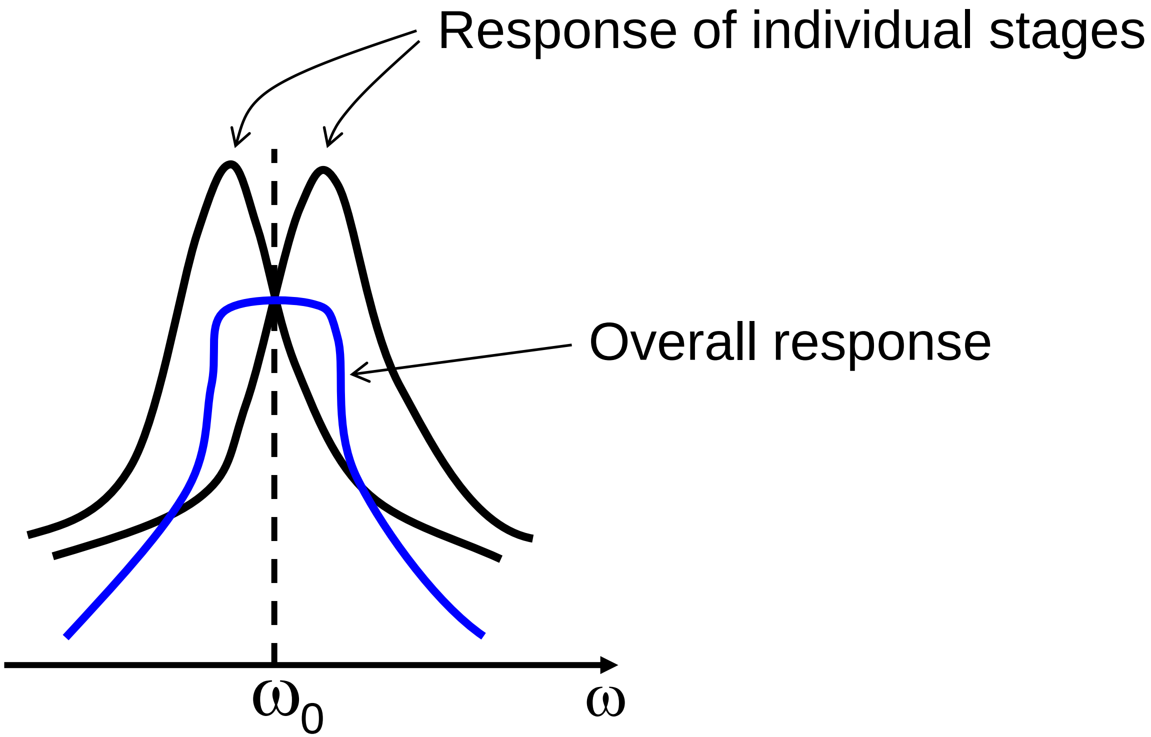
where $\sqrt{2^{1/N} - 1}$ is known as the bandwidth-shrinkage factor



Frequency response of a synchronously tuned amplifier.

Stagger-Tuning

- Maximal flatness around f_0 (Center frequency)

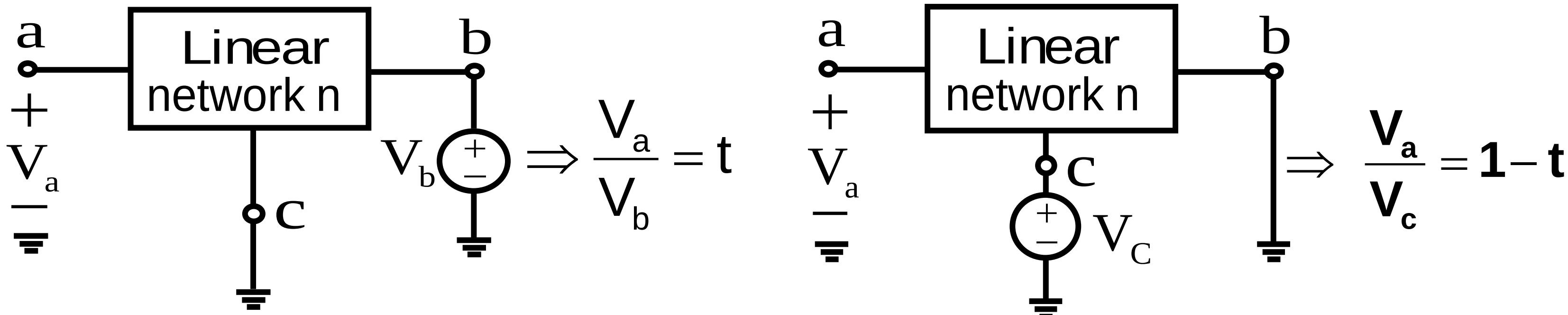


Appendix

- Complementary transformation
- Sensitivity
- Stagger-Tuning

Complementary Transformation

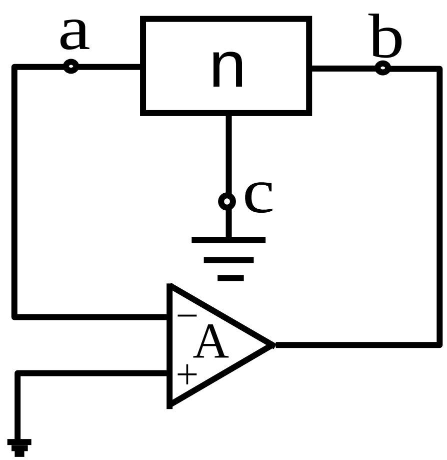
- Complement of transfer function
  - Interchanging input and ground

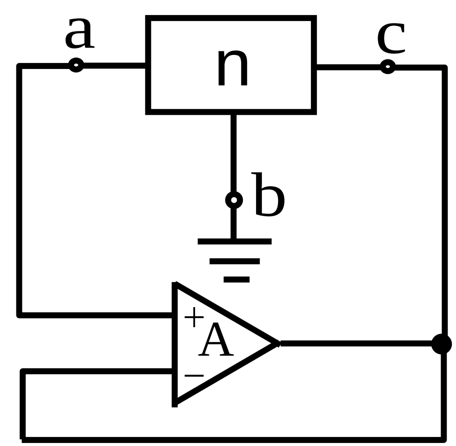


- Two-step procedure
  1. Nodes of the feedback network and any of the op amp inputs that are connected to ground should be disconnected from ground and connected to the op amp output. Conversely, those nodes that were connected to the op amp output should be now connected to ground. That is, we simply interchange the op amp output terminal with ground.
  2. The two input terminals of the op amp should be interchanged.

Complementary Transformation (Cont.)

- Example





characteristic equation

$$1 + At = 0$$

characteristic equation

$$1 - \frac{A}{1+A}(1-t) = 0$$

same poles

$$\Rightarrow \; A + 1 - A(1-t) = 0$$

$$\Rightarrow \; 1 + At = 0$$

Sensitivity

- Real components deviate from their designed values
  - Initially inaccurate due to fabrication tolerances
  - Drift due to environmental effects such as temperature and humidity
  - Chemical changes which occurs as the circuit ages
  - Inaccuracies in modeling the passive and active devices, e.g., nonideal OPAMP and parasitics
- All coefficients, and therefore poles and zeros of H(s), depend on circuit element
- The size of H(s) error depends on how large the component tolerances are and how sensitive the circuit’s performance is to these tolerances

Sensitivity (Cont.)

- Sensitivity calculation, allow the designer
  - to select the better circuit from those in the literature
  - to determine whether a chosen filter circuit satisfies and will keep satisfying the given specifications
- Component χ
- Performance criterion p(χ), such as
  - Quality factor
  - Pole frequency
  - Zero frequency

or p(s,χ), if p is also a function of frequency and stands for

1. H(s), or
2. magnitude of H(s), or
3. phase of H(s)

Sensitivity (Cont.)

- Sensitivity S_{χ}^{P}
  - Taylor series

$$P(s,\chi) = P(s,\chi_0) + \left.\frac{\partial P(s,\chi)}{\partial \chi}\right|_{\chi_0} d\chi + \frac{1}{2}\left.\frac{\partial^2 P(s,\chi)}{\partial \chi^2}\right|_{\chi_0} (d\chi)^2 + \ldots\ldots$$

$$\text{if } \frac{d\chi}{\chi_0} << 1 \text{ and } \left.\frac{dp}{d\chi}\right|_{\chi=\chi_0} \text{ is small}$$

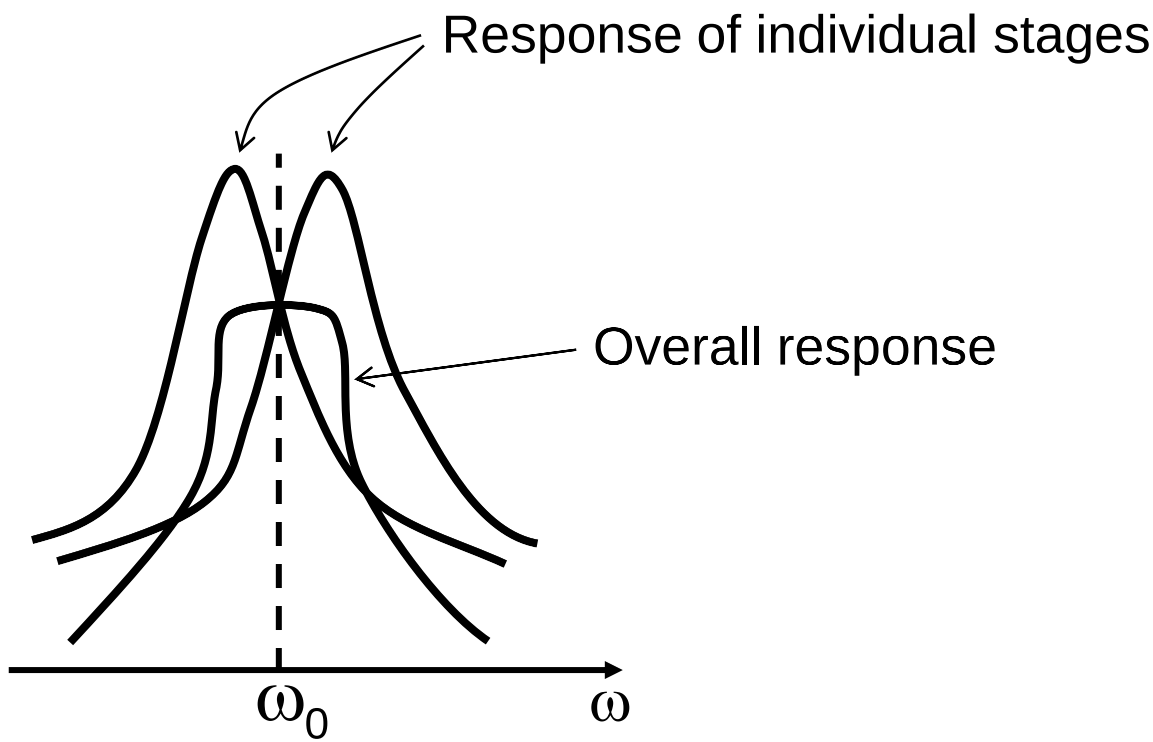
$$\Rightarrow \begin{cases} \Delta P(s,\chi_0) = P(s,\chi_0 + d\chi) - P(s,\chi_0) \approx \left.\frac{\partial P(s,\chi)}{\partial \chi}\right|_{\chi_0} d\chi \\ \frac{\Delta P(s,\chi_0)}{P(s,\chi_0)} \approx \frac{\chi_0}{P(s,\chi_0)} \left.\frac{\partial P(s,\chi)}{\partial \chi}\right|_{\chi_0} \frac{d\chi}{\chi_0} \end{cases}$$

$$\Rightarrow S_{\chi}^{P} = \frac{\chi_0}{P(s,\chi_0)} \left.\frac{\partial P(s,\chi)}{\partial \chi}\right|_{\chi_0} = \left.\frac{\partial P / P}{\partial \chi / \chi}\right|_{\chi_0} = \left.\frac{d(\ln P)}{d(\ln \chi)}\right|_{\chi_0}$$

$$\text{if } \frac{\partial \chi}{\chi_0} << 1, \text{ then } S_{\chi}^{P} \approx \frac{\Delta P / P}{\Delta \chi / \chi}$$

Stagger-Tuning

- Maximal flatness around f_0 (Center frequency)



Stagger-Tuning (Cont.)

- Narrow-band approximation
  - For a second order

$$T(s)=\frac{a_1 s}{s^2+s\frac{\omega_0}{Q}+\omega_0^2}=\frac{a_1 s}{(s+\frac{\omega_0}{2Q}-j\omega_0\sqrt{1-1/4Q^2})(s+\frac{\omega_0}{2Q}+j\omega_0\sqrt{1-1/4Q^2})}$$

For $\begin{cases} Q>>1 \text{ (narrow-band filter)} \\ \text{values of s in the neighborhood of } +j\omega_0 \end{cases}$

$$T(s)\approx\frac{a_1 s}{(s+\frac{\omega_0}{2Q}-j\omega_0)(s+\frac{\omega_0}{2Q}+j\omega_0)}\approx\frac{a_1/2}{s+\frac{\omega_0}{2Q}-j\omega_0}=\frac{a_1/2}{(s-j\omega_0)+\frac{\omega_0}{2Q}}$$

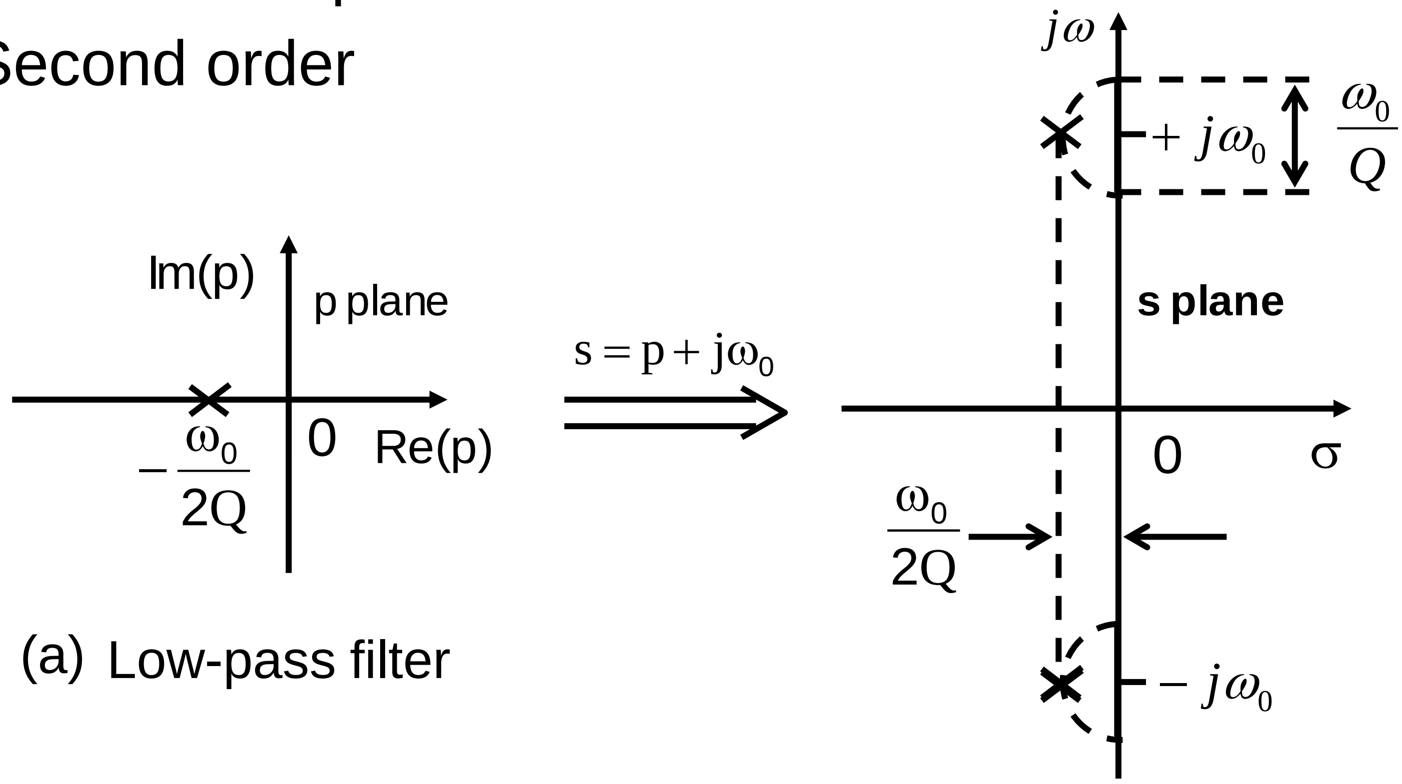
  - Frequency transformation

Let $p=s-j\omega_0$

$\Rightarrow s=p+j\omega_0$

Low-Pass to Band-Pass Transformation

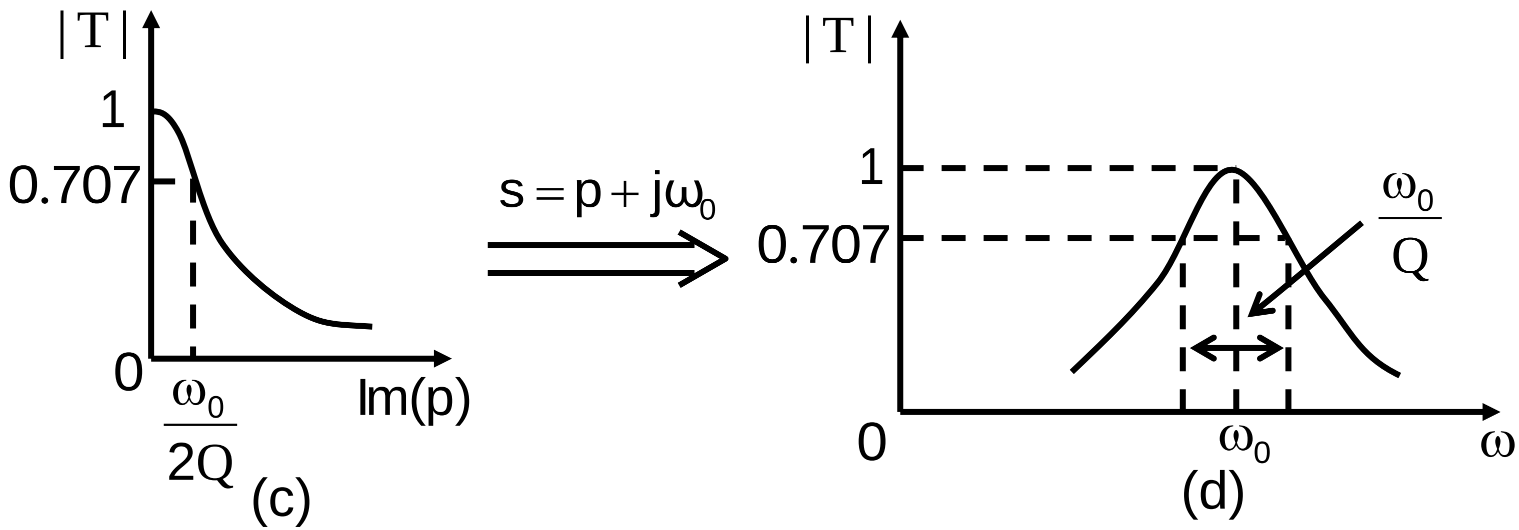
- Maximally flat response can be obtained by transforming Butterworth low-pass filters
  - Second order



(a) Low-pass filter

(b) Band-pass filter

Low-Pass to Band-Pass Transformation (Cont.)



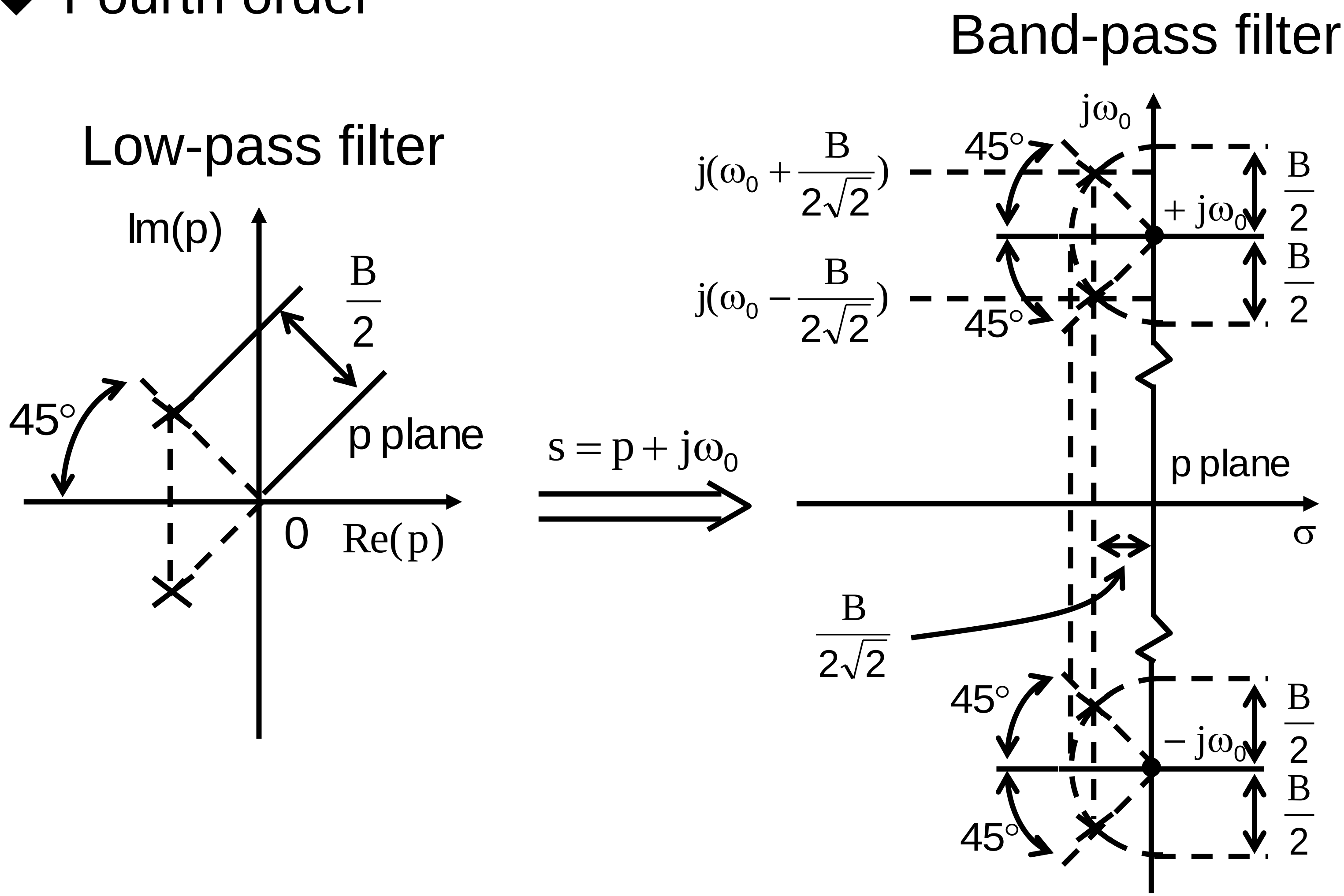
$$T(p) = \frac{k}{p + \frac{\omega_0}{2Q}} \quad \xrightarrow{s = p + j\omega_0} \quad T(s) = \frac{a_1/2}{(s - j\omega_0) + \frac{\omega_0}{2Q}}$$

⇨ Narrow-bond approximation

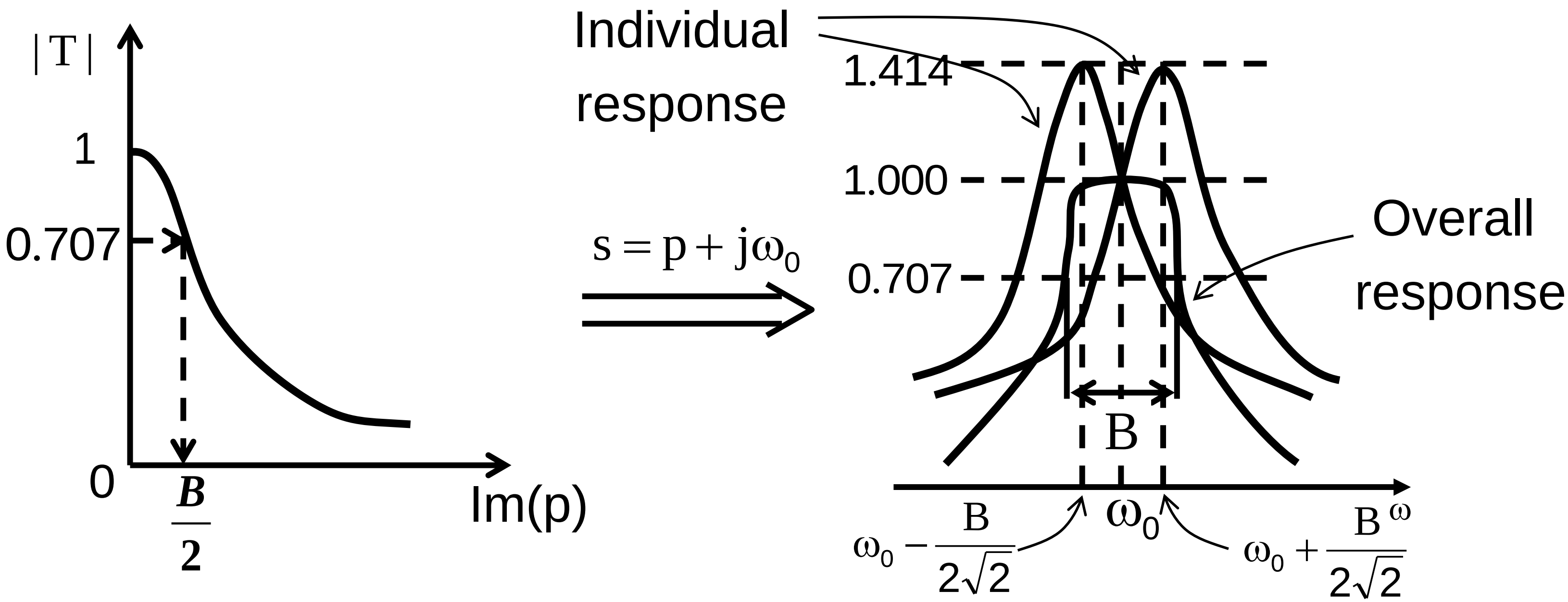
(refer to previous page 11-65)

Low-Pass to Band-Pass Transformation (Cont.)

- Fourth order



Low-Pass to Band-Pass Transformation (Cont.)



- Obtaining the poles and the frequency response of a fourth-order stagger-tuned narrow-band band-pass amplifier by transforming a second-order low-pass maximally flat response.

$$\omega_{01} = \omega_0 + \frac{B}{2\sqrt{2}},\ B_1 = \frac{B}{\sqrt{2}},\ Q_1 \approx \frac{\sqrt{2}\omega_0}{B},\quad \omega_{02} = \omega_0 - \frac{B}{2\sqrt{2}},\ B_2 = \frac{B}{\sqrt{2}},\ Q_2 \approx \frac{\sqrt{2}\omega_0}{B}$$